Problems

8.1. On a particular day the orographic cloud on the island of Hawaii is 2 km thick with a uniform liquid water content of 0.5 g/m$^3$. A drop of 0.1 mm radius at cloud top begins to fall through the cloud.

(a) Find the size of the drop as it emerges from cloud base, neglecting vertical air motions in the cloud. In this and subsequent parts of the problem, neglect growth by condensation, use the elementary form of the continuous-growth equation, and assume a collection efficiency of unity.

(b) Assuming that the terminal velocity of the drop is equal to $k_3 r$, where $k_3 = 8 \times 10^3$ s$^{-1}$, find the time taken by the drop to fall through the cloud.

(c) Hawaiian orographic clouds are actually maintained by gentle upslope motions, which cause a steady, weak updraft. Solve for the size of the drop in part (a) as it emerges from the cloud if there is a uniform updraft of 20 cm/s.
8.2 To allow for ventilation effects, the diffusional growth equation is sometimes approximated by

$$r \frac{dr}{dt} = (S - 1) \xi_1 (1 + 0.3 \text{Re}^{1/2}),$$

where $\xi_1 = [F_k + F_d]^{-1}$ and Re is the Reynolds number characterizing the flow around the drop of radius $r$. Assess the importance of ventilation by comparing the time required for a drop to evaporate completely using (1) the approximation including the ventilation factor and (2) the same equation neglecting this factor. Make the comparison for drops with initial radii of 1 mm, 0.5 mm, and 0.1 mm. For environmental conditions assume $T = 10^\circ C$, $p = 80$ kPa, and a relative humidity of 50%. Approximate the drop fall velocity by the linear law of problem 8.1.
So the growth equation becomes

\[ \frac{dr}{dt} = (S-1) S_1 (1 + Kr) \]

where

\[ S_1 = 10^{2.10} = 100 \text{ (um)}^2 \text{ s}^{-1} = 10^{-10} \text{ m}^3 \text{ s}^{-1} \]

\[ T = 10^6 \text{C}, \ p = 80 \text{ kPa} \text{ from Fig. 7.1} \]

\[ K = 0.3 \sqrt{\frac{2\rho R_g}{\mu}} \]

\[ \mu = 1.766 \times 10^{-5} \times (\frac{160}{80}) \text{ kg m}^{-1} \text{ s}^{-1} = 2.21 \times 10^{-5} \text{ kg m}^{-2} \text{ s}^{-1} \]

\[ \rho d = \frac{p}{RT} = \frac{80 \times 10^2}{287 \times 283} \approx 0.98 \text{ kg m}^{-3} \]

\[ K = 0.3 \sqrt{\frac{2 \times 0.98 \text{ kg m}^{-3} \times 8 \times 10^3 \text{ s}^{-1}}{2.21 \times 10^{-5} \text{ kg m}^{-2} \text{ s}^{-1}}} \approx 0.9 \times 10^4 \text{ m}^{-1} \]

So

\[ \frac{dr}{dt} = A (1 + Kr), \text{ where } A = (S-1) S_1 = 0.5 \times 10^{-10} \text{ m}^3 \text{ s}^{-1} \]

\[ \int_{r_0}^{r_1} \frac{dr}{1 + Kr} = \int_{t_0}^{t_1} A \ dt = AT_1 \]

\[ \text{LHS} = \int_{r_0}^{r_1} \left[ \frac{1}{K} \left( 1 - \frac{r_0}{1 + Kr} \right) \right] dr \]

\[ = \left[ \frac{r}{K} - \frac{r_0}{K^2} \ln(1 + Kr) \right]_{r_0}^{r_1} = -\frac{r_0}{K} + \frac{1}{K^2} \ln(1 + Kr) \]

Thus,

\[ T_1 = \frac{1}{A} \left[ -\frac{r_0}{K} + \frac{1}{K^2} \ln(1 + Kr) \right] \]

where \( A = (S-1) S_1 = 0.5 \times 10^{-10} \text{ m}^3 \text{ s}^{-1} \)

\[ K = 0.9 \times 10^4 \text{ m}^{-1} \]

But if neglecting the ventilation effect, the growth equation becomes

\[ \frac{dr}{dt} = (S-1) S_1 = A \]

\[ \int_{r_0}^{r_2} A \ dt = AT_2 \Rightarrow \frac{r_0^2}{2} = AT_2 \]

\[ \Rightarrow T_2 = \frac{-\frac{r_0^2}{2A}}{A} \]

where \( A = (S-1) S_1 = 0.5 \times 10^{-10} \text{ m}^3 \text{ s}^{-1} \)
Consider the ventilation effect:

\[ r_0 = 1 \text{ mm } (= 10^{-3} \text{ m}) \]

\[ T_1 = \frac{1}{(-0.5 \times 10^{-10} \text{ m}^2 \text{s}^{-1})} \left\{ \frac{(1 \times 10^{-3} \text{ m})^2}{0.9 \times 10^5 \text{ m}^2 \text{s}^{-1}} + \frac{\mu L}{(0.9 \times 10^5 \text{ m}^2 \text{s}^{-1})^2} \right\} \approx 1652.5 \approx 27.5 \text{ min} \]

\[ r_0 = 0.5 \text{ mm } (= 5 \times 10^{-4} \text{ m}) \]

\[ T_1 = \ldots \approx 692.5 \approx 11.5 \text{ min} \]

\[ r_0 = 0.1 \text{ mm } (= 1 \times 10^{-4} \text{ m}) \Rightarrow T_1 = \ldots \approx 64.5 \approx 1.07 \text{ min} \]

Neglecting the ventilation effect:

\[ r_0 = 1 \text{ mm } (= 10^{-3} \text{ m}) : \]

\[ T_2 = - \frac{(1 \times 10^{-3} \text{ m})^2}{2 \times (-0.5 \times 10^{-10} \text{ m}^2 \text{s}^{-1})} = 10^4 \text{ s} \approx 16.7 \text{ min} \]

\[ r_0 = 0.5 \text{ mm } (= 5 \times 10^{-4} \text{ m}) : \]

\[ T_2 = - \frac{(5 \times 10^{-4} \text{ m})^2}{2 \times (-0.5 \times 10^{-10} \text{ m}^2 \text{s}^{-1})} = 2500 \text{ s} \approx 41.7 \text{ min} \]

\[ r_0 = 0.1 \text{ mm } (= 1 \times 10^{-4} \text{ m}) \]

\[ T_2 = - \frac{(1 \times 10^{-4} \text{ m})^2}{2 \times (-0.5 \times 10^{-10} \text{ m}^2 \text{s}^{-1})} = 100 \text{ s} \approx 1.67 \text{ min} \]
8.3. Neglecting ventilation effects, calculate the distance a drop of 0.1 mm radius falls while evaporating completely under the conditions of problem 8.2. Compare the solution based on the linear fall speed approximation with a solution using the data of Gunn and Kinzer (Table 8.1).
8.4. A drop of 0.2 mm diameter is inserted in the base of a cumulus cloud that has a uniform liquid water content of 1.5 g/m³ and a constant updraft of 4 m/s. Using the elementary form of the continuous-growth equation and neglecting growth by condensation, determine the following:

(a) the size of the drop at the top of its trajectory;
(b) the size of the drop as it leaves the cloud;
(c) the time the drop resides in the cloud.

Assume a collection efficiency of unity, and for the dependence of fall velocity on size use the data in Table 8.1. (Note: Parts (a) and (b) of this problem are well suited for graphical solution.)

8.5. One member of a population of cloud droplets, all 10 μm in radius, grows by condensation and coalescence. Assume a cloud water content of 1 g/m³ and a supersaturation of 0.2 %, and solve for the time it takes for the drop to reach 20, 30, and 40 μm radius. For simplicity, assume that the droplets remain 10 μm in size. Use the continuous-growth equation for coalescence, allowing for the size and fall speed of the droplets. For the fall speed use Stokes’ Law.
Allow for the collection efficiency of the drop relative to the droplets using the data in Table 8.2, and assuming a coalescence efficiency of unity. This problem should be solved graphically.

8.6. A small drizzle drop is swept upwards in a cumulus congestus cloud and grows by accretion and condensation in the supersaturated environment. The condensation parameter $\xi$ may be regarded as constant and the linear fall speed law of problem 8.1 approximates the relative velocity between the growing drop and the cloud droplets. Develop and solve the differential equation that describes the growth of the drop by accretion and condensation acting simultaneously. Compare the result with the approximation obtained by adding the solutions for growth by accretion and by condensation acting separately.
\[ \text{LHS} = \int_{r_0}^{r_i} \frac{r \, dr}{\sqrt{r^2 + br^2}} = \frac{1}{2b} \ln \left( \frac{\sqrt{r^2 + br^2}}{\sqrt{r_0^2 + br_0^2}} \right) = \frac{1}{2b} \ln \left( \frac{\sqrt{r_i^2 + br_i^2}}{\sqrt{r_0^2 + br_0^2}} \right) \]

\[ \Rightarrow \quad \ln \left( \frac{\sqrt{r_i^2 + br_i^2}}{\sqrt{r_0^2 + br_0^2}} \right) = 2bt, \quad \frac{r_i^2 + br_i^2}{\sqrt{r_0^2 + br_0^2}} = e^{2bt} \]

or
\[ r_i^2 = (l_0^2 + \frac{5}{b}) e^{2bt} - \frac{5}{b} \]  \hfill (D)

(2) Consider each effect separately,

for condensation growth only \( \Rightarrow \) \( \frac{dr}{dt} = \frac{5}{b} \Rightarrow r = r_0 - 2.5t \)

for accretion growth only \( \Rightarrow \) \( \frac{dr}{dt} = br \Rightarrow \int\frac{dr}{r} = \int b \, dt \Rightarrow \ln(r) = bt \)

\[ \Rightarrow r = r_0 e^{bt} \quad \text{or} \quad r = r_0 e^{2bt} \]

Adding both solutions together:

\[ r^2 = r_0^2 + 2.5r + r_0 e^{2bt} \]  \hfill (E)

(D) - (E):

\[ r_i^2 - r_2^2 = \frac{3}{b} e^{2bt} - \frac{5}{b} - r_0^2 - 2.5t \]

\[ \approx \frac{5}{b} (1+6bt) - \frac{5}{b} - r_0^2 - 2.5t \quad \text{(if } bt<<1\text{)} \]

\[ = -r_0^2 < 0 \]

So initially \( \text{when } t << \frac{1}{b} \Rightarrow r_i < r_2, \) i.e., the radius of drop if considering both effects simultaneously is smaller than the radius of drop if considering both effects separately.

After some time \( \text{when } t >> \frac{1}{b} \Rightarrow r_i > r_2, \) i.e., the radius of drop if considering both effects simultaneously is larger than the radius of drop if considering both effects separately.
8.7. Prove that cloud droplets move very quickly in response to changes in the ambient air velocity, by solving for the effective time constant characterizing the response. Determine the relationship between this time constant and the droplet terminal fall velocity.

Starting from (8.2) and assuming very small Reynolds number \( \frac{C_{D}Re}{24} \approx 1 \)

\[ F_{R} = 6 \pi \mu r u \]  
\( (8.2) \)

The vertical equation of motion is

\[
\begin{align*}
\begin{aligned}
\frac{du}{dt} + \frac{9 \mu}{2 \pi r^{2}}u &= \frac{g}{m} \\
\frac{d\rho}{dt} + \frac{9 \mu}{2 \pi r^{2}} \rho &= \frac{g}{m}
\end{aligned}
\end{align*}
\]

Integration factor for the above equation is 
\[ \eta = \exp \left[ -\int \frac{g}{2 \pi r^{2}} dt \right] = e^{-\frac{gt}{2 \pi r^{2}}} \]

So the ODE becomes

\[
\int \left[ u(t) e^{\frac{9 \mu t}{2 \pi r^{2}}} \right] dt = \int \frac{g}{m} e^{\frac{9 \mu t}{2 \pi r^{2}}} dt
\]

\[
\left[ u(t) e^{\frac{9 \mu t}{2 \pi r^{2}}} \right]_{u(0)}^{u(t)} = \frac{g}{m} \left[ e^{\frac{9 \mu t}{2 \pi r^{2}}} \right]_{0}^{\infty} = \frac{g}{m} \left[ e^{\frac{9 \mu t}{2 \pi r^{2}}} - 1 \right]
\]

\[
\Rightarrow u(t) e^{\frac{9 \mu t}{2 \pi r^{2}}} - u(0) = \frac{g}{9 \mu} \left[ e^{\frac{9 \mu t}{2 \pi r^{2}}} - 1 \right]
\]

Thus

\[
\begin{align*}
\begin{aligned}
u(t) &= u(0) e^{\frac{9 \mu t}{2 \pi r^{2}}} + u_{e} \left[ 1 - e^{-\frac{9 \mu t}{2 \pi r^{2}}} \right] \\
\text{where } u_{e} &= \frac{2 g \rho r^{2}}{9 \mu}
\end{aligned}
\end{align*}
\]

The effective time constant \( T \) is

\[
T = \frac{2 g \rho r^{2}}{9 \mu} = \frac{u_{e}}{g}
\]

(For terminal velocity at time \( t \to \infty \)