Tropical Cyclogenesis via Convectively Forced Vortex Rossby Waves in a Three-Dimensional Quasigeostrophic Model

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ABSTRACT

This work investigates the problem of tropical cyclogenesis in three dimensions. In particular, the authors examine the interaction of small-scale convective disturbances with a larger-scale vortex circulation in a nonlinear quasigeostrophic balance model. Convective forcing is parameterized by its estimated net effect on the potential vorticity (PV) field. Idealized numerical experiments show that vortex intensification proceeds by ingestion of like-sign potential vorticity anomalies into the parent vortex and expulsion of opposite-sign potential vorticity anomalies during the axisymmetrization process. For the finite-amplitude forcing considered here, the weakly nonlinear vortex Rossby wave mean-flow predictions for the magnitude and location of the spinup are in good agreement with the model results. Vortex development is analyzed using Lagrangian trajectories, Eliassen–Palm flux vectors, and the Lorenz energy cycle.

Using numerical estimates of the magnitude of PV injection based on previous observational and theoretical work, the authors obtain spinup to a 15 m s$^{-1}$ cyclone on realistic timescales. Simulation of a midlevel vortex with peripheral convection shows that axisymmetrization results in the spinup of a surface cyclone. The axisymmetrization mechanism demonstrates the development of a warm-core vortex. The relative contribution from eddy-heat and eddy-momentum fluxes to the warm core structure of the cyclone is investigated.

The vortex spinup obtained shows greater than linear dependence on the forcing amplitude, indicating the existence of a nonlinear feedback mechanism associated with the vortex Rossby waves.

Building on recent work by several authors, this work further clarifies the significance of the axisymmetrization process for the problem of tropical cyclogenesis. The theory is shown to be consistent with published observations of tropical cyclogenesis. Further observational and modeling tests of the theory, specific to the dynamics examined here, are proposed.

1. Introduction

Tropical cyclones form in the presence of convective disturbances in the Tropics. Many such disturbances are present in an ocean basin at any given time. However, only a small fraction (about 80 per year) evolve into tropical storms, which may in turn become hurricanes. It is generally accepted that favorable climatological conditions for tropical cyclogenesis include the presence of low-level convergence, the presence of low-level cyclonic relative vorticity, the absence of strong vertical shear of the horizontal winds, an atmosphere conducive to deep moist convection, a significant value of the planetary vorticity, and sea surface temperatures greater than approximately 26°C with a deep oceanic mixed layer. About 80% of all tropical cyclones originate in or near the monsoon troughs or the intertropical convergence zone; most others form from disturbances in the easterly trade winds (Gray 1968; Frank 1987).

The cyclogenesis process examined here builds on the studies of vortex axisymmetrization and vortex Rossby wave dynamics by MacDonald (1968), Guinn and Schubert (1993), Smith and Montgomery (1995), Kallenbach and Montgomery (1995), and Montgomery and Kallenbach (1997). Guinn and Schubert made an extensive study of the connection between PV or vortex Rossby waves and mature hurricane spiral bands, including a simulation of the potential vorticity dynamics of convective asymmetries. Montgomery and Kallenbach further clarified the structure of radially and azimuthally propagating Rossby waves on a circular vortex, and also examined vortex Rossby wave–mean-flow interactions in a quasi-linear nondivergent model. They hypothesized that initial vorticity asymmetries such as those that arise from moist convective forcing would accelerate the mean tangential winds and proposed this interaction as a mechanism for tropical cyclogenesis. These ideas are analogous to the problem of the intensification and maintenance of a large-scale zonal jet by forced planetary Rossby waves (Shepherd 1987), except the pertinent vorticity gradient in our case is the radial vorticity gradient of the vortex. This paper tests the Montgomery–Kallenbach vortex spinup mech-
anism in a fully nonlinear, three-dimensional quasigeostrophic balance model. The three-dimensional character of the model allows the investigation of the vortex's eddy-forced secondary circulation, crucial to understanding the origin of the vortex's warm core.

We examine the problem of vortex development associated with the production of cyclonic vertical vorticity by cumulus convection. Such convection could be caused by environmental forcing (Challa and Pfeffer 1980, 1990; Pfeffer and Challa 1981; Montgomery and Farrell 1993; Challa et al. 1998) or by mesoscale processes. Although it has long been known that energy extracted from the underlying ocean and realized as latent heat during condensation provides the principal energy source for tropical cyclones, and this process has been simulated by numerical models such as that of Ooyama (1969) and Kurihara and Tuleya (1981), this work further elucidates the nonaxisymmetric advective dynamics that promotes upscale transfer of convective-scale energy to vortex-scale energy. Organization of convection by the storm is not necessary in our model, although organization into a tropical cloud cluster or mesoscale convective system is assumed. The cyclogenesis mechanism proposed here does not require a cooperative interaction between convection and the large-scale vortex. Cooperative interaction (Smith 1997; Stevens et al. 1997; Ooyama 1982) could enhance the development process presented here at later stages in the cyclone life cycle and may become the dominant mechanism by the hurricane stage. For simplicity, we also ignore the possible effects of the ambient vertical shear.

Our work describes cyclogenesis from a preexisting mesoscale vortex in the presence of convection, such as that which would be present in a tropical cloud cluster. The initiating vortex in our simulations has a 200-km radius of maximum tangential winds (RMW) and an initial maximum tangential wind speed of 5 m s$^{-1}$. The physical setup is consistent with typical values for mesoscale convectively generated vortices (Johnston 1981; Bartels and Maddox 1991). Cyclonic vortices of this type could also be found within the closed flow patterns occurring in easterly waves (Reed et al. 1977). PV anomalies induced by cumulus convection on the periphery of the vortex assimilate into the parent vortex via axisymmetrization. Although this process can be described phenomenologically as vortex merger and stripping, we show that it is also useful to characterize it in terms of the interaction between vortex Rossby waves and the mean flow. The wave–mean-flow approach, which has proven successful in describing other atmospheric phenomena such as the quasi-biennial oscillation and sudden stratospheric warmings (Holton 1992), is shown here to yield valid quantitative predictions for the magnitude and location of the vortex spinup at the finite amplitudes considered relevant to the cyclogenesis problem. The formation of a 5-K warm core within a reasonable timescale provides compelling evidence that our proposed mechanism may capture the essence of the cyclogenesis process. In addition, we find that symmetric outbreaks of penetrative convection near the center of the preexisting vortex will give stronger and more rapid spinup.

Sensitivity studies of the axisymmetrization mechanism reveal the presence of nonlinear feedback in the spinup process, with the increase in tangential velocity generally having a greater than linear proportionality to the forcing amplitude.

The cyclogenesis mechanism proposed here may be contrasted with cyclogenesis due to the interaction and merger of mesoscale convective vortices (MCVs) produced in the stratiform region of mesoscale convective systems in a favorable large-scale environment (Simpson et al. 1997; Ritchie and Holland 1997; Ritchie 1995). Our mechanism focuses on the process by which PV (or vorticity) anomalies due to moist penetrative convection relax to axisymmetry in the presence of a preexisting vortex, such as an MCV or a low-level cyclonic vorticity anomaly. The difference between these two approaches may lie in the emphasis placed on the importance of convective versus stratiform heating. The two ideas are not mutually exclusive; in fact, Ritchie and Holland (1997; section 4b) describe an interaction of two low-level circulations in the early stages of cyclogenesis that may well be an example of our process. In addition, our work demonstrates the usefulness of the underlying vortex Rossby wave dynamics in describing the redistribution of convectively induced PV.

In section 2 we summarize the models used in our studies. Section 3 describes the basic states and initializations used. In section 4 we examine three-dimensional vortex axisymmetrization in the presence of convection. Section 5 examines vortex spinup in the presence of ongoing convection and describes sensitivity tests of our results. Section 6 reviews the most relevant observations of tropical cyclogenesis to our theory. Section 7 summarizes the results of our work and proposes further observational and modeling tests of the theory.

2. The numerical models

a. The quasigeostrophic and semispectral models

The chief model employed is a three-dimensional Boussinesq quasigeostrophic model on an f plane. Here f is the constant Coriolis parameter and, for simplicity, the static stability $N^2$ is assumed uniform throughout the troposphere. Henceforth $Q$ denotes the total quasi-geostrophic PV, $\theta$ the flow potential temperature (total minus resting state), $\zeta$ the relative vertical vorticity, and $\phi$ the flow geopotential. Here, $\theta_s$ is the surface potential temperature, $H$ the depth of the model troposphere, and $L$ the Rossby radius $NHf = 1000$ km. For convenience, all numerical calculations are performed in nondimensional units. Denoting nondimensional variables with tildes we let...
Table 1. Multiplication factors to be used in converting parameters from nondimensional to dimensional units.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal distance (x, y, z)</td>
<td>(10^6) m</td>
</tr>
<tr>
<td>Vertical distance (z)</td>
<td>(10^4) m</td>
</tr>
<tr>
<td>Time (t)</td>
<td>(10^1) s</td>
</tr>
<tr>
<td>Time period (T)</td>
<td>(\tau_{eddy} = 2.91) days</td>
</tr>
<tr>
<td>Horizontal velocity (u, v)</td>
<td>(100) m s(^{-1})</td>
</tr>
<tr>
<td>Vertical velocity (w)</td>
<td>(1) m s(^{-1})</td>
</tr>
<tr>
<td>Temperature ((\theta))</td>
<td>(30) K</td>
</tr>
<tr>
<td>Geopotential ((\phi))</td>
<td>(10^4) J kg(^{-1})</td>
</tr>
<tr>
<td>Potential vorticity (Q)</td>
<td>(0.3 \times 10^{-4}) m(^2) s(^{-1}) K kg(^{-1})</td>
</tr>
</tbody>
</table>

\[(x, y, z, t) = (\tilde{x}L, \tilde{y}L, \tilde{z}H, \tilde{t}f^{-1}), \quad (1)\]

\[(u_x, v_y, w) = NH \left( \tilde{u}_x, \tilde{v}_y, \frac{H}{L} \tilde{w} \right), \quad (2)\]

\[(\phi, \theta, Q) = \left( \tilde{\phi}N^2H^2, \tilde{\theta}N^2H_0, \tilde{Q}f \right). \quad (3)\]

Here \((x, y, z)\) are the Cartesian coordinates with \(z\) the pseudoheight (Hoskins and Bretherton 1972), \(t\) the time, and \((u_x, v_y, w)\) the zonal geostrophic, meridional geostrophic, and vertical velocities, respectively. We also define \((r, \lambda)\) to be the cylindrical coordinates with \(r = 0\) at the center of the initial azimuthal mean vortex on the lowest level and \(u\) and \(v\) to be the radial and tangential geostrophic velocities.

In the model simulations we use \(f = 10^{-4}\) s\(^{-1}\) and \(H = 10\) km. Although the value for \(f\) is roughly twice that found at tropical latitudes, we have chosen the higher value in order to keep the initial Rossby number \(R\) of the symmetric vortex less than unity. Here \(R = \frac{\bar{v}}{fr}\), where \(\bar{v}\) is the azimuthal-mean tangential wind. The justification for using the larger value of \(f\) is based on the idea of incorporating the average rotation rate of an incipient vortex into the definition of \(f\). The justification behind this idea is provided in Shapiro and Montgomery (1993), where the balance theory derived was based on a generalized Rossby number incorporating the local rotation rate of the vortex. An analysis of our results using the primitive equations and symmetric balance equations (Shapiro and Montgomery 1993) is a topic of current work, the results of which will be reported in due course. Note that the basic asymmetric results described in section 4 have maximum Rossby numbers of approximately 0.5 with Rossby numbers of about 0.3 at the RMW. The quasigeostrophic approximation is used here because of its simplicity; we believe that its regime of practical validity extends beyond the range of formal validity \(R \ll 1\).

In the quasigeostrophic Boussinesq approximation, the PV conservation equation, the invertibility equation, and the thermodynamic equation in nondimensional form are, respectively,

\[
\frac{DQ}{Dt} \hat{\xi} = 0, \quad (4)
\]

\[
\hat{\nabla}^2 \hat{\phi} + \frac{\hat{\nabla}^2}{\hat{\nabla}^2} = Q - 1, \quad (5)
\]

and

\[
\frac{DQ}{Dt} + \hat{\xi} + w = 0, \quad (6)
\]

where tildes have been dropped, \(\hat{\nabla}^2\) is the horizontal Laplacian operator and \(DQ/Dt\) is the material derivative operator following the geostrophic wind. Henceforth, only nondimensional variables are used unless specified. The numerical solution of Eqs. (4), (5), and (6) is summarized in appendix A.

The fully nonlinear barotropic wavenumber two simulation described in section 4a was performed with a semispectral model based on the two-dimensional nondivergent barotropic vorticity equation on an \(f\) plane. Details of this model are given in appendix B.

b. Physical description of convection in the quasigeostrophic model

Although it is obviously not possible to represent convective-scale dynamics (e.g., Weisman et al. 1993; Trier et al. 1997) with a quasigeostrophic model, the approach taken here is a phenomenological one whereby the vertical vorticity budget of an ensemble of convective cells near or within an incipient vortex is parameterized by potential vorticity anomalies having a horizontal scale of approximately 200 km. Neglecting internal friction, Ertel’s PV equation based on dry potential temperature is

\[
\frac{DQ}{Dt} = \hat{\xi} \cdot \hat{\nabla} \hat{\theta},
\]

where \(\hat{\xi}\) is the absolute vorticity, \(\rho\) is the density, and \(\hat{\theta}\) is the heating rate associated with cumulus convection. In the quasigeostrophic Boussinesq approximation, only the vertical component of \(\hat{\xi}\), \(\hat{\nabla} \hat{\theta}\) is assumed significant, and the density is treated as constant, yielding

\[
\frac{DQ}{Dt} = (\hat{\xi} \cdot \hat{\nabla}) \frac{\partial \hat{\theta}}{\partial z}. \quad (7)
\]

Although the strict quasigeostrophic approximation
would neglect $\zeta$ compared to $f$ in Eq. (7), we retain both components of the vorticity for these numerical estimates.

Since realistic convective heating profiles have $\partial \theta / \partial z$ greater than zero at low levels and less than zero at high levels, convection is seen to create a positive PV anomaly at low levels and a negative PV anomaly aloft. Since $\theta$ is assumed zero at the horizontal boundaries (i.e., no explicit enthalpy fluxes at the ocean surface), no PV is fluxed into the domain and, consequently, the mass-weighted integral of PV will not change (Hoskins et al. 1985; section 7).

The magnitudes of the convectively induced PV anomalies can be determined if the heating rate is known. Convective heating rates for midlatitude convective systems have been studied both observationally and theoretically (Gallus and Johnson 1991; Hertenstein 1996); an appropriate average of $\theta$ averaged over several convective cells is found to be $\theta_{\text{max}} = 15$ K h$^{-1}$ or 360 K d$^{-1}$. From recent radar observations of tropical mesoscale convective systems (Mapes and Houze 1995) one obtains an estimate for $\theta_{\text{max}}$ of 190–380 K d$^{-1}$. Assuming further that the heating rate depends on $z$ as $\sin(\pi z/H_c)$, where $H_c$ is the vertical scale, and that the incipient vortex has relative vorticity $O(f)$, we find

$$\text{PV change} \approx \int_0^{6h} \frac{2f\theta_{\text{max}}}{\rho} \frac{\pi}{H_c} dt' \approx 1.2 \text{ PVU},$$

where one PVU $= 10^{-6}$ m$^2$ s$^{-1}$ K kg$^{-1}$. Here we have used $f = 5 \times 10^{-5}$ s$^{-1}$, $H_c = 15$ km, and taken $\rho$ to be 1.0 kg m$^{-3}$ for a realistic estimate of the magnitude of PV generation in the lower troposphere. As suggested by the observations of Zehr (1992) to be discussed in section 5 and displayed in Fig. 16, we integrate for a period of 6 h corresponding to the bursts of convection shown in the figure. Although the peaks in the figure have a duration of approximately 12 h, we expect the lifetime of the cold cloud tops to be longer than the time period during which heating is occurring.

The main vortex’s PV can be estimated similarly:

$$\text{PV}_{\text{main vortex}} \approx \frac{(f + \zeta) \partial \theta_{\text{total}}}{\rho} \frac{\pi}{\partial z} \approx \frac{2f\theta_{\text{max}}N^2}{g\rho} \approx 0.3 \text{ PVU}.$$ 

Thus, even being quite conservative, we see that the convective anomalies for our system can have roughly the same PV magnitude as the basic-state vortex.

3. Basic states and initialization

a. Introduction

To lay the conceptual foundation for the dynamics of ongoing moist convection in vortex shear flow, we study first the free adjustment or “relaxation” associated with finite-amplitude asymmetric PV anomalies near the radius of maximum winds of an incipient cyclonic vortex. The combination of a basic-state barotropic circular vortex with three different asymmetries is analyzed. The three representative examples using the initial barotropic vortex are a barotropic wavenumber 2 anomaly and three-dimensional two-cluster and single-cluster anomalies that are intended to represent the net effect of tropical convection. The two-cluster and single-cluster convective anomalies are defined in section 5c. For our fourth example we consider the problem of a midlevel vortex in the presence of convection.

b. The basic state vortices

For our first three examples the initial flow consists of a barotropic circular vortex superposed with either of three types of perturbation asymmetries. The initial circular vortex for the barotropic simulation is the same as that used in the nondivergent calculations of Montgomery and Kallenbach (1997; section 2). The basic-state circular vortex for the three-dimensional model is defined by the PV field

$$Q_{\text{b}}(x, y, z) = 1 + \alpha_0 e^{-\rho_0 r^2},$$

(8)

where

$$r^2 = (x - x_c)^2 + (y - y_c)^2,$$

(9)

$(x_c, y_c)$ is the center of the vortex, and parameters such as $\alpha_0$ and $\beta_0$ are listed in Table 2. For both cases the basic-state vortex has an RMW of 0.2 (200 km) and a maximum tangential wind of 0.05 (5.0 m s$^{-1}$). Future work should investigate the precise thresholds of basic-state vortex strength and areal extent required for the cyclogenesis process described here to occur. The maximum pressure drop for the basic state in the three-dimensional model is 2.7 mb. Figure 1 shows the corresponding radial profiles of the azimuthal-mean PV, azimuthal-mean vorticity, azimuthal-mean flow geopotential, and azimuthal-mean tangential wind for the three-dimensional model basic state. In the three-dimensional model, to ensure compatibility with the doubly periodic boundary conditions, the basic state PV has been adjusted so that the area integral of the anomalous basic-state PV is zero. This adjustment is a small effect and amounts to the requirement that the net circulation is zero on each vertical level.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0$</td>
<td>0.900</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>32.0</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>100</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>100</td>
</tr>
<tr>
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<tr>
<td>$x_2$</td>
<td>0.75</td>
</tr>
<tr>
<td>$y_2$</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Fig. 1. Radial profiles of the azimuthal mean potential vorticity, absolute vertical vorticity, flow geopotential, and tangential wind for the basic-state vortex used in the quasigeostrophic model. To obtain PV in PVU (10^{-2} m^2 s^{-1} K kg^{-1}), multiply by 0.3. To obtain vorticity in s^{-1}, multiply by 10^{-2}. To obtain geopotential in J kg^{-1}, multiply by 10^4. To obtain tangential wind in m s^{-1}, multiply by 100. To obtain radial displacement in km, multiply by 1000.

For our fourth example we consider the case of a basic-state vortex that, rather than being barotropic, has maximum winds at the middle z level. This configuration is of meteorological interest because, as noted in section 1, our initial basic-state vortex could be a mesoscale convectively generated vortex; MCVs typically have maximum tangential winds at midlevels (Johnston 1981; Bartels and Maddox 1991). To model such an initial flow we take

$$Q_{\text{midlevel}} = 1 + \alpha_0 e^{-\beta_0 r_0^2} \sin(\pi z),$$

where $r_0^2 = (x - x_c)^2 + (y - y_c)^2$ with the vortex center $(x_c, y_c)$, the same as for the barotropic vortex. The parameters $\alpha_0$ and $\beta_0$ are listed in Table 2 and are the same as were used for the barotropic basic-state vortex.

c. Initial PV anomalies

For the nondivergent simulation, the basic-state vortex is perturbed with a localized wavenumber 2 vorticity disturbance whose Fourier amplitude is given by

$$\tilde{\zeta}_2(r) = \begin{cases} 0.20 \zeta (\text{RMW}) \sin^2 \left( \frac{\pi (r - 0.070)}{0.27} \right), & |r - \text{RMW}| \leq 0.13; \\ 0, & |r - \text{RMW}| > 0.13. \end{cases}$$

Here $\zeta (\text{RMW})$ is the azimuthal mean vorticity at the vortex’s RMW and the parameters are chosen to give a maximum asymmetry at the RMW, where the physical space asymmetry amplitude is 40% of the basic-state amplitude. A map plot of the relative vorticity for this anomaly added to the basic state is shown in the initial condition of Fig. 2.

The second and third asymmetries considered were used in the three-dimensional model to simulate the effects of an outbreak of cumulus convection near the initially circular vortex. A PV anomaly that has approximately the same magnitude as the basic-state vortex PV and has the desired property of adding positive PV at low levels and depleting it at upper levels while keeping the mass-weighted integral of PV unchanged is given by

$$Q_2(x, y, z) = \alpha_2(e^{-\beta_2 (\delta r_1)^2} + e^{-\beta_2 (\delta r_2)^2}) \cos(\pi z),$$

where $(\delta r_1)^2 = (x - x_{1c})^2 + (y - y_{1c})^2$, $(\delta r_2)^2 = (x - x_{2c})^2 + (y - y_{2c})^2$, and the parameters $\alpha_2$, $\beta_2$, $x_{1c}$, $x_{2c}$, $y_{1c}$, and $y_{2c}$ are listed in Table 2. This configuration, called the two-cluster convective anomaly, has two regions of convection on opposite sides of the basic-state vortex at radii of 0.25 (250 km) from the center of the vortex. Contour plots of the PV, vorticity, and flow geopotential for $Q_2 = Q_{\text{basic}} + Q_1$ are shown in Fig. 3. The PV anomalies are slightly larger at maximum than the basic-state vortex maximum.\(^1\) The maximum pressure drop of each positive anomaly associated with $Q_2$ is 1.0 mb, with a corresponding maximum tangential wind velocity of 3.0 m s^{-1}. The maximum temperature deviation from the resting basic state for occurs at $z = 0.5$ and is approximately 0.75 K. (The pressure, temperature, and wind fields for the $Q_2$ anomalies by themselves are not shown.)

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\(^1\) Thus, on the 200-km scale the associated heating rate is taken to be a factor of 3–6 smaller than the observational estimate of section 2b.
Fig. 2. Evolution of the nondimensional relative vertical vorticity and the relative vertical vorticity asymmetry for the barotropic wavenumber 2 initial condition. Only the inner 800 km × 800 km of the model domain is shown. Time is nondimensional with $T = 1$ corresponding to one eddy turnover time (see Table 1 for details).
As a model for convection that is localized in one area, we take
\[ Q'_3 = \alpha_3 e^{-\beta_3(\text{Re})^2} \cos(\pi z) \]  
and refer to this configuration as the single-cluster convective anomaly. Plots of the PV, vorticity, and flow geopotential for \( Q'_3 = Q_{\text{conv}} + Q'_3 \) are shown in Fig. 4. The maximum pressure drop of the positive anomaly associated with \( Q'_3 \) is 0.92 mb, with a corresponding maximum tangential wind of 2.8 m s\(^{-1}\). The maximum temperature deviation from a resting basic state for the anomaly occurs at \( z = 0.5 \) and is approximately 0.75 K. (The pressure, temperature, and wind fields for the \( Q'_3 \) anomaly are not shown.)

When the negative anomalies associated with \( Q'_3 \) and \( Q'_1 \) are added to the basic-state PV the total PV remains positive, thereby ensuring symmetric stability (\( JQ > 0 \)) of the model atmosphere.

For the midlevel vortex example we used the same single-cluster convective anomaly as that described above. Thus the PV distribution for this example is given by \( Q_4 = Q_{\text{midlevel}} + Q'_3 \).

4. Fundamentals of three-dimensional vortex axisymmetrization

a. The barotropic wavenumber two asymmetry

To validate the wave–mean flow predictions of Montgomery and Kallenbach (1997), as well as to provide a conceptual foundation for the upcoming baroclinic experiments, we consider first the relaxation of the initial barotropic wavenumber 2 anomaly. Figure 2 shows the evolution of the total and asymmetric relative vorticity for this case. Waves propagating both azimuthally and radially are clearly evident in the asymmetric vorticity.
In particular, as the positive and negative vorticity perturbations are sheared into trailing spirals by the mean vortex the perturbations propagate outward. Excitation of secondary wave features in the interior of the vortex following the shearing of the initial asymmetries is also evident.

The axisymmetrizing wave disturbances evident in Fig. 2 cannot be gravity waves since gravity waves are completely excised in the nondivergent model. Indeed, the waves are vortex Rossby waves whose restoring mechanism is associated with the radial gradient of basic-state vortex vorticity. The basic theory for these waves was developed by Montgomery and Kallenbach (1997). The radial group velocity for vortex Rossby wave packets in the WKB approximation is given by

\[ C_{gt} = \frac{-2kn\overline{\zeta_0}}{R_0 \left( k^2 + \frac{n^2}{R_0^2} \right)} \]

where \( n \) is the azimuthal wavenumber and \( \overline{\zeta_0} \) is the basic-state radial vorticity gradient at reference radius \( r = R_0 \). The radial wavenumber \( k \) is given by \( k(t) = k_0 - n\overline{\Omega_0} \), where \( k_0 \) is its initial value, \( t \) is time, and \( \overline{\Omega_0} \) is the radial gradient of the basic-state angular velocity. Outside the RMW, our initially circularly symmetric anomaly will be deformed to a downshear-tilted patch. For downshear tilt, \( k \gg 0 \). Since for our vortex \( \overline{\Omega} < 0 \), \( k \) will grow more positive with time. If the vorticity gradient is negative, as is initially the case, \( C_{gt} \) will be positive and wave packets will propagate outward. In the upcoming examples using larger-amplitude initial asymmetries, the outermost wave creates a region of positive mean vorticity gradient; in that region, \( C_{gt} \ll 0 \) and wave packets propagate inward.

The solid curves in Fig. 5 show the wave-induced changes in azimuthal mean relative vorticity and tangential velocity predicted by the nonlinear semispectral model. The dashed lines show the corresponding quasi-
linear prediction obtained by using just the linear solution to evaluate the eddy vorticity flux. Overall, the agreement is quite good. A notable feature of Fig. 5 is the acceleration or spinup of the mean tangential winds and mean vorticity near the radius of the initial asymmetry. This spinup was predicted by Montgomery and Kallenbach (1997); as we will see, it persists in the three-dimensional setting.

Although significant excitation of higher wavenumber components does occur, analysis of the wavenumber 2 vorticity shows that the instantaneous wavenumber 2 packet propagates radially outward to a stagnation radius ($r = 0.27$) close to the zero in the $\delta \zeta$ plot. This qualitative behavior was predicted in the WKB framework of Montgomery and Kallenbach (1997) and is consistent with the wave activity interpretation of Held and Phillips (1987) generalized to vortex flow. Quantitatively, the quasilinear prediction for $\delta \zeta_{\text{max}}$ is $0.10$ m s$^{-1}$; the value observed in the nonlinear model is $0.09$ m s$^{-1}$. The Montgomery and Kallenbach (1997) results, valid at second order in the wave amplitude, are still approximately valid. Thus the interaction of vortex Rossby waves with the mean flow captures the essence of the physics at these small but finite amplitudes. At higher amplitudes we find that wave–wave interactions play a more significant role, but the structure of $\delta \zeta$ and $\delta \zeta$ are nevertheless observed to be qualitatively similar to their small amplitude counterparts (e.g., see Fig. 9).

As the initial asymmetry is sheared by the mean vortex, some of the positive vorticity anomaly is transported toward the interior of the vortex. The remaining portion of the positive vorticity anomaly is transported outward to form the positive vorticity filaments that orbit the vortex core. The negative vorticity anomaly is also transported outward and becomes more nearly axisymmetric. Lagrangian trajectory analyses (see appendix C) lend further support to these ideas about vorticity transport in the symmetrization process. A contour plot (not shown) of the initial azimuthal-mean vorticity subtracted from the total vorticity at $T = 2.5$ exhibits central regions of vorticity augmentation surrounded by spiral arms of vorticity depletion, with an outer region of vorticity augmentation as well. The authors studied Lagrangian back-trajectories for particles destined to be found in each of these regions. Particles in the inner vorticity augmentation region were found to originate from the initial positive vorticity anomalies, in agreement with our understanding that positive vorticity from the anomalies moves inward and pools near the center of the vortex. Particles in the vorticity depletion region backtrack to the initial vorticity depletion regions of the wavenumber 2 anomaly.

A nearly neutral discrete vortex Rossby mode associated with a sign change of the radial vorticity gradient is visible in the barotropic experiment for $T \geq 4$. Such modes are discussed in section 4g.

\subsection*{b. The two-cluster convective anomaly}

We consider next the two-cluster convective anomaly. Figure 6 shows the potential vorticity field for top, middle, and bottom $z$ surfaces as a function of time. At the top level, the negative anomalies are expelled from the main vortex and are advected around the vortex. The negative anomalies are not assimilated by the vortex but remain intact. At low $z$ levels the behavior is similar to the barotropic case discussed above; the positive anomalies merge into the main vortex, with accompanying production of high-PV filaments, and again the vortex symmetrizes almost completely. However, due to the wave-induced change in the mean vorticity gradient shown below, neutral or weakly unstable modes are evident in the inner core region (see section 4g).

Figure 7 shows the azimuthal mean and asymmetric...
PV as a function of time at the highest model level \( (z = 1) \). Disturbances to the mean flow are evident. As noted earlier, the PV deficit regions move outward. Axisymmetrization does not occur at upper levels. The flow there resembles a vortex tripole (Polvani and Carton 1990; Orlandi and van Heijst 1992). From a linear wave viewpoint the tripole can be thought of as a finite amplitude generalization of a wavenumber 2 discrete or weakly unstable vortex Rossby mode that is supported by a sign change in the mean radial PV gradient (see section 4g).

The bottom level (Fig. 8) shows an increase in mean PV at a radius of approximately 0.1 (100 km), especially at earlier times. As in the barotropic case, there is an overall steepening of the PV gradient. Figures 9a,b show the changes in the azimuthally averaged PV and tangential velocity at the bottom \( z \) surface. The similarity to the barotropic example (Fig. 5) is striking. The pooling of high PV at radius 0.125 (125 km) is evident. A spinup of 1.0 m s\(^{-1}\) occurs over the period of \( 1T_{\text{eddy}} \) (approximately 3 days). No further acceleration is observed after \( 1T_{\text{eddy}} \).

In the double-cluster simulation, the stagnation radius of the initial wavenumber 2 wave packet is again approximately equal to the radius of the zero in the \( \delta \vec{V} \) plot \( (r = 0.3) \). Taking the primary azimuthal mode and its next two harmonics into account, the value of \( \delta T_{\text{max}} \) estimated by the quasi-linear nondivergent model is 0.94 m s\(^{-1}\).

Forward Lagrangian trajectories for the two-cluster convective case at the lowest model level have also been analyzed. Particles were placed randomly on a circle of radius 0.1 (approximately the size at half-maximum of the positive anomaly) centered at the anomaly center. The results for a few selected particles are shown in Fig. 9c. Many of the particles move inward toward \( r = 0.125 \) (125 km). The others move outward to form the outer high-PV filaments. Trajectories at the highest level (not shown) were also analyzed. At this level, particles from the negative anomalies move generally...
Fig. 7. Evolution of the azimuthal mean PV and asymmetric PV for the two-cluster convective anomaly at the highest model level ($z = 1$). Contour interval is fixed at 0.1 for all plots. One nondimensional unit of radial displacement equals 1000 km.
Fig. 8. Evolution of the azimuthal mean PV and asymmetric PV for the two-cluster convective anomaly at the lowest model level ($z = 0$). One nondimensional unit of radial displacement equals 1000 km.
outward, in agreement with Fig. 7. Particles from the positive anomalies either move inward to form the slight bump of positive mean PV shown at \( r \approx 0.225 \) in the radial profiles of Fig. 7, or move outward to form the secondary maximum at \( r \approx 0.45 \).

The Lorenz energy cycle (Holton 1992) for the two-cluster convective experiment has also been examined (Enagonio and Montgomery 1998). As expected, the largest energy storage is found to be in the mean-flow kinetic energy; the largest energy conversion is that from eddy kinetic energy to mean kinetic energy, corresponding to the spinup of the basic-state vortex by the eddies observed with the wave–mean flow diagnostics.

c. The single-cluster convective anomaly

For our third example we consider the single-cluster convective anomaly. The single-cluster configuration is intended to simulate a localized outbreak of convection near the preexisting vortex. For this case the center of the \((r, \lambda)\) coordinate system is taken to be the instantaneous geopotential minimum on the lowest level. The behavior is in many ways similar to the two-cluster convective anomaly. At the top level, the positive and negative anomalies move apart; at the lowest level the two positive anomalies move together and symmetrize. The PV evolution for the single-cluster case at the bottom level is shown in Fig. 10.
Fig. 10. Evolution of the azimuthal mean PV and asymmetric PV for the single-cluster convective anomaly at the lowest model level ($\zeta = 0$). Contour interval is fixed at 0.1 for all plots. One nondimensional unit of radial displacement equals 1000 km. Radial profiles extend only to approximately 700 km because of the motion of the lowest-level center of the system.
Fig. 11. (a) Change in azimuthal mean PV ($\delta Q$) and (b) change in azimuthal mean-tangential wind ($\delta \bar{v}$) at $T = 2.5$ for the single-cluster convective case at the lowest model level ($z = 0$). Here, $\delta \bar{v}$ in $\text{m s}^{-1}$ is obtained by multiplying by 100. Radial displacement in km is obtained by multiplying by 1000. (c) Forward trajectories for the single-cluster convective case at the lowest-model level ($z = 0$). The dotted line shows the location of the anomaly from which the particles originate. Only the inner 800 km × 800 km of the model domain is shown.

The lowest-level $\delta Q$ and $\delta \bar{v}$ distributions, plotted in Figs. 11a,b, show a new phenomenon: transport of PV to the center of the vortex. This occurs because the single-cluster convective anomaly has a wavenumber 1 Fourier component; only wavenumber 1 can transport particles to the center of the vortex. The Lagrangian trajectories show the same effect. Figure 11c shows the forward trajectory paths for a few selected particles originating on the $+\text{PV}$ anomaly for the single-cluster case. Unlike Fig. 9c, in which particles are excluded from the center of the vortex, in Fig. 11c particles pass arbitrarily close to the center.

Figure 11b shows a low-level spinup of about 0.4 m s$^{-1}$ for single-cluster convection. This is about a factor of 2.5 smaller than for the two-cluster convective anomaly. This reflects the fact that the total initial forcing in the single-cluster case is approximately a factor of 2 smaller; we use the same magnitude anomaly in both cases, but there are two positive anomalies for the two-cluster case and only one for the one-cluster example.

For a comparison of the spinup due to single-cluster versus double-cluster anomalies as a function of the anomaly amplitude, see section 4f.

Table 3 shows the dependence of the final surface maximum azimuthal mean-tangential wind speed about the geopotential minimum on the radial location of the convective patch for the single-cluster convective configuration. Convection at or near the center of the basic-
state vortex produces significantly greater values of $\bar{\nabla}_{\text{max}}$. For radii much greater than the initial radius of maximum winds of the basic-state vortex, a slight spin-down ($\delta \bar{\nabla}_{\text{max}} < 0$) is observed. The final radius of maximum winds also depends on the initial radial location of the convection, ranging from approximately 150 km for convection at the center to 235 km for convection at 500-km radius.

The largest increase in tangential velocity occurs when the convective patch is placed at the center of the initial basic-state vortex. In view of this fact, the reader may wonder why our emphasis is on the role of asymmetries in the spinup process. However, it should be noted that, assuming that at the genesis stage the pre-existing vortex does not play a strong role in organizing the convection, it is more probable that convective outbreaks will occur somewhere near the periphery of the vortex rather than very close to the center; thus we believe that asymmetric processes are likely to be an important contribution to mechanisms for tropical cyclogenesis. As discussed in section 6, observations appear to indicate that asymmetries play an important role in the genesis process. When symmetric convection does occur, it can be expected to yield stronger and more rapid spinup.

d. A midlevel vortex with convection

The time evolution of the midlevel vortex configuration is summarized in Figs. 12 and 13. The figures show contours of PV on $z = 0$, $z = 0.25$, $z = 0.5$, $z = 0.75$, and $z = 1$, as well as contours of PV versus $x$ and $y$ on $z = 0$. The low-level PV anomaly, initially at 250 km from the center of the basic-state vortex, is drawn approximately 200 km into the center to a position underneath the main vortex and subsequently remains aligned with it, producing a vertically stacked vortex in the mid- to low-level troposphere. In contrast, the negative PV anomaly at upper levels is expelled laterally from the axis of the now-aligned vortex system. This example demonstrates that our axisymmetrization mechanism could cause an MCV to appear to “build downward” in the presence of peripheral convection, resulting in spinup of strong winds at the surface.

The midlevel vortex experiment shows some spinup of the azimuthal-mean tangential wind about the center of the midlevel vortex. At $T = 0$, the mean-tangential wind maximum occurs at about 300 km from the axis of the mean vortex, approximately corresponding to the position of the convective anomaly. The surface mean tangential wind maximum measured about the axis of the midlevel vortex at the surface is 2.3 m s$^{-1}$. At $T = 2.5$, the RMW at the surface has contracted to 150 km, and the mean-tangential wind maximum has increased to 3.0 m s$^{-1}$, with a maximum spinup of 1.7 m s$^{-1}$. The surface vortex is now capable of further growth if convection continues to occur around it.

<table>
<thead>
<tr>
<th>Radius of convective anomaly (km)</th>
<th>$\bar{\nabla}_{\text{max}}$ (m s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7.0</td>
</tr>
<tr>
<td>100</td>
<td>6.4</td>
</tr>
<tr>
<td>250</td>
<td>5.6</td>
</tr>
<tr>
<td>400</td>
<td>4.9</td>
</tr>
<tr>
<td>500</td>
<td>4.7</td>
</tr>
</tbody>
</table>

e. Warm core formation through axisymmetrization

In the course of the two-cluster relaxation experiment, a slight warming of 0.08 K in the azimuthal mean-temperature field is observed at mid- to low levels. As we will see in section 5, in the presence of ongoing convection a strong warm core of magnitude 3–5 K forms during the axisymmetrization process. We now discuss the physical basis for the warming observed in the relaxation experiments.

On azimuthally averaging the thermodynamic equation, we obtain

$$\frac{\partial}{\partial t} \frac{\partial \Phi}{\partial r} = -\frac{\partial}{\partial r} \left[ ru \frac{\partial \Phi}{\partial z} \right] - N^2 \bar{\nabla} + \frac{\partial}{\partial t} \bar{\nabla}.$$

Contributions to $\partial \Phi / \partial t$ come from radial eddy-heat flux, mean vertical motion, and heating due to convection. In the relaxation experiments after the initial pulse $\dot{\theta} = 0$.

The mean vertical motion is given by the radial derivative of the transverse streamfunction:

$$\bar{\Psi} = \frac{\partial}{\partial r} \bar{\Psi},$$

where $\bar{\Psi}$ is deduced upon solving the Sawyer–Eliassen equation

$$N^2 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \bar{\Psi} \right) + f^2 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial z} \bar{\Psi} \right) = \frac{\partial E}{\partial r} - \frac{\partial F}{\partial z},$$

with

$$E = -\frac{\partial}{\partial r} \left( \frac{ru}{r} \frac{\partial \Phi}{\partial z} \right),$$

and

$$F = -\frac{f}{r} \frac{\partial}{\partial r} [u(r \bar{\nabla} \bar{\Psi})].$$

Note that Eq. (13) is dimensional. Here, $E$ represents the contribution from eddy-heat flux forcing and $F$ represents the contribution from eddy-momentum flux forcing.

Neglecting for the moment the contribution from the eddy-heat fluxes, the physics responsible for the mean vertical motion can be illustrated with a simple example. We consider the change in the mean tangential wind $\delta \bar{\nabla}$
Fig. 12. Contours of PV versus $x$ and $y$ on $z = 0$, $z = 0.25$, $z = 0.5$, $z = 0.75$, and $z = 1$, as well as a plan view of contours of PV versus $x$ and $y$ on $z = 0$ for the midlevel vortex with single-cluster convection at time $T = 0.002$ days.

Fig. 13. Contours of PV versus $x$ and $y$ on $z = 0$, $z = 0.25$, $z = 0.5$, $z = 0.75$, and $z = 1$, as well as a plan view of contours of PV versus $x$ and $y$ on $z = 0$ for the midlevel vortex with single-cluster convection at time $T = 7.07$ days.
induced by eddy-momentum flux forcing following the imposition of a convective PV anomaly at low levels near the undisturbed circular vortex. As we have already seen in section 4a, a local torque is exerted on the mean circular vortex during the axisymmetrization process that causes a net spinup of the tangential winds. Because the secondary circulation always acts to oppose changes induced by the geostrophic flow, the mean-transverse circulation consists of a radially outward flow near the \( \delta \tau \) maximum that tends to resist the spinup. Similarly, near the \( \delta \tau \) minimum the mean-transverse circulation is radially inward. The resultant mean convergence leads to mean ascent in between the two regions; by mass conservation, mean subsidence develops near the center of the vortex. When the contribution from eddy-heat fluxes is included the dynamics is no longer so simple.

As a first look at the relative importance of heat and momentum flux contributions to the development of the warm core, azimuthal mean Eliassen–Palm (EP) flux vectors (Edmon et al. 1980) for the two-cluster relaxation experiment have been analyzed (Enagonio and Montgomery 1998). In cylindrical coordinates, the quasigeostrophic EP flux vector is defined by

\[
\mathbf{F} = \left(-r \mathbf{u}' \mathbf{u}' - \frac{f}{N^2 r l_0^2} \frac{\partial \phi'}{\partial z}\right)\mathbf{r}.
\]

A plot of the EP flux vectors along with contours of the radial PV flux (\( u'Q' \)) shows that at low levels for \( r < 0.3 \), \( u'Q' \) is negative, indicating that the eddy PV flux is upgradient (into the vortex); for \( r > 0.3 \), \( u'Q' \) is positive, showing the downgradient flux of eddy PV into the high-PV filaments at larger radii. The plot also indicates that at this time the heat flux is into the vortex at upper levels, out of the vortex at midlevels, and negligible at low levels. At low levels (\( z < 0.4 \)) the cyclonic momentum flux is predominantly inward, and greater values of \( \mathbf{\nabla} \cdot \mathbf{F} \) here agree with our other findings that the energy transferred to the mean flow is greater at lower levels.

As indicated in Eq. (13), the mean-transverse streamfunction is derived from derivatives of the fluxes rather than from the fluxes themselves. For a direct comparison of the effects of momentum flux to heat flux we turn to explicit computation of the forcing terms in the Sawyer–Eliassen equation. A simulation initialized with the two-cluster convective anomaly was studied with resultant fields output every 0.125\( \tau _{\text{steady}} \) in order to study the rapid azimuthal shearing of the asymmetry. At each output time the two forcing terms \( \partial \mathbf{E}/\partial r \) and \( -\partial \mathbf{F}/\partial z \) were calculated and plotted. The results for some selected times are shown in Fig. 14. We see from Fig. 14 that at early times the momentum flux term dominates the heat flux term everywhere. The effect of the momentum flux is best seen by looking at \( -\partial \mathbf{F}/\partial z \) at \( T = 0.375 \) in Fig. 14. The momentum flux forcing at low levels inside the radius of maximum winds is strongly positive. The elliptic operator in Eq. (13) tends to reverse the sign of the operand, so that in the region where \( -\partial \mathbf{F}/\partial z \) is strongly positive, \( \delta \tau \) is strongly negative. It follows that \( \delta \tau \) is negative near the center of the vortex at low levels; thus there is subsidence in that region as expected.

All throughout this process the heat flux term is non-zero; by \( T = 0.375 \) the heat flux becomes a significant contribution to the forcing at upper levels. Nevertheless, the maximum warming occurs in the mid- to lower levels where the momentum flux is dominant.

\[ f. \text{Nonlinear feedback in the axisymmetrization process} \]

As will be evident in the section on applications to tropical cyclogenesis, it is of interest to study the dependence of the maximum spinup \( \delta \tau _{\text{max}} \) on the asymmetry amplitude. In the quasi-linear nondivergent regime \( \delta \tau \) scales as \( \eta ^2 \tau _{\text{max}} \) where \( \tau _{\text{max}} \) is the maximum tangential wind of the basic-state vortex and \( \eta \) is the strength of the asymmetry relative to the basic-state vortex (Montgomery and Kallenbach 1997). Using \( \eta \sim Q'/Q_{\text{max}} \), where \( Q' \) is the maximum of the PV anomaly and \( Q_{\text{max}} \) is the maximum PV of the basic state, and recalling that the invertibility relation is linear, we find that \( \delta \tau \sim Q'/Q_{\text{max}} \). In our experiments \( Q'/Q_{\text{max}} \) is not small, but this scaling may nevertheless be approximately valid. Experiments were performed to test the dependence of \( \delta \tau _{\text{max}} \) and the maximum temperature change \( \Delta T_{\text{max}} \) on \( Q' \) and \( \tau _{\text{max}} \).

Figure 15 shows the results of these tests, along with the predicted maximum \( \delta \tau \) and maximum \( \delta T \) assuming linear and quadratic scaling in \( Q' \). Note that, since the upper levels do not become axisymmetric, one obtains small fluctuations in the azimuthal mean quantities for the single-cluster experiments as the center of the system moves. For the two-cluster case at the tested amplitudes, the scaling of \( \delta \tau _{\text{max}} \) is intermediate between linear and quadratic. However, the temperature change \( \Delta T_{\text{max}} \) has a greater than quadratic dependence on \( Q' \). For the single-cluster case, the dependence of \( \delta \tau _{\text{max}} \) is greater than quadratic, but the variation of the temperature change with amplitude is quite small.

The dependence of \( \delta \tau _{\text{max}} \) and \( \Delta T_{\text{max}} \) on \( \tau _{\text{max}} \) was also tested. For the double-cluster experiment, \( \delta \tau _{\text{max}} \) is found to be approximately inversely proportional to \( \tau _{\text{max}} \) for an increase of 20% in \( \tau _{\text{max}} \). For the single-cluster case, an increase of 20% in \( \tau _{\text{max}} \) results in a 24% decrease in \( \delta \tau _{\text{max}} \). It is interesting to note that for both cases, \( \Delta T_{\text{max}} \) is essentially unaffected by a change in vortex strength. We have not yet developed a scaling argument that predicts the dependence of \( \Delta T_{\text{max}} \) on anomaly amplitude and vortex strength.

These results are indicative of nonlinear feedback in the vortex dynamics. From the dimensional vertical vorticity equation for quasigeostrophic dynamics,
Fig. 14. Momentum flux and heat flux forcing terms for the Sawyer–Eliassen equation as a function of time for the two-cluster convective relaxation experiment. To obtain the forcing terms in $s^{-1}$, multiply the nondimensional quantity by $10^{-13}$. 

Fig. 15. Maximum $\delta \bar{v}$ and maximum change in temperature as a function of anomaly amplitude for the double-cluster and single-cluster relaxation experiments. Model results are compared with predictions assuming linear and quadratic scaling. The amplitude is displayed in units of the nominal anomaly pulse amplitude. Here, $\delta \bar{v}$ is in m s$^{-1}$ and $\delta \bar{T}$ in K.}

\[ \frac{\partial \xi_r}{\partial t} = -\frac{1}{r \, \partial r} (ru' \xi_r) - f \frac{\partial}{\partial r} (r \Pi_u) \]  

(14)

some type of nonlinear spinup effect is expected.

On the face of it one might claim that the spinup in these experiments occurs simply because the azimuthal mean-radial secondary circulation is converging the convectively generated relative vorticity into the storm’s inner core. This idea is incorrect for two reasons. First, as seen from Eq. (14), in quasigeostrophic theory the azimuthal mean circulation (which is solely ageostrophic) converges only planetary, not relative vorticity. Sec-
ond, most of the spinup occurs in the inner core of the storm where the azimuthal mean-radial circulation at low levels is outward (hence divergent). From the PV viewpoint, since potential vorticity is advected solely by the geostrophic wind, its advection into the core occurs purely through eddy transports.

\( g. \) Amplitude sensitivity and wave-induced discrete vortex modes

In all the experiments described thus far, the upper-level PV anomalies do not become axisymmetric. In addition to the fact that the upper-level anomalies exist in favorable shear, we believe that the disruption of axisymmetrization at upper levels can be traced to the existence of discrete neutral or unstable vortex Rossby modes propagating azimuthally around the vortex center. For linear waves neutral nonsingular modes require a vanishing radial PV gradient somewhere in the flow (Pedlosky 1987, section 7.8), while unstable modes require \( \delta q/\delta r \) to change sign at least once (Gent and McWilliams 1986). To investigate these ideas, we studied smaller-amplitude disturbances that permitted us to enter a quasi-linear regime.

When the amplitude of the anomalies was decreased by a factor of 5 from the nominal case, we still obtained a weak wave-induced sign change of the azimuthal radial PV gradient at upper levels near \( r = 350 \) km. Under these conditions one can discern a persistent upper-level wavenumber 2 disturbance after the axisymmetrization at low levels is complete. Despite the presence of differential rotation at all levels, the disturbance retained its shape as it propagated cyclonically around the vortex.

In order to verify the fact that this mode’s existence depended on the sign change of the azimuthal mean radial PV gradient, we moved the anomalies inward to a position 150 km from the center of the vortex. This change had the effect of superposing the anomalies on a larger value of the basic-state PV so that the basic-state \( \delta q/\delta r \) dominated the wave-induced radial PV gradient. In this configuration we observed axisymmetrization at all levels, verifying the disappearance of the discrete or weakly unstable baroclinic modes.

The existence of wave-induced discrete neutral or weakly unstable baroclinic vortex modes raises intriguing possibilities about their ability to orchestrate further convective bursts near the RMW and sustain intensification. A thorough investigation of these ideas requires a self-consistent cumulus convection model, which is beyond the scope of this paper. This topic, as well as further investigation of the underlying dynamics of the discrete vortex Rossby modes, remains for future work.

5. Three-dimensional vortex dynamics under convective forcing: Application to tropical cyclogenesis

\( a. \) Pulsed convective studies

In the previous section, we described studies of the axisymmetrization of a vortex forced by initial asymmetric PV anomalies. We examined various aspects of this problem and demonstrated that the PV anomalies cause vortex spinup.

For a more realistic model of cyclogenesis, we should incorporate the fact that convection is not simply an initial forcing condition on the vortex; convective activity is often ongoing. Figure 16 (from Zehr 1992) shows a time series of convective activity in a tropical cloud cluster that eventually became Typhoon Abby in 1983. The level of convection is indicated by the fraction of the cloud cluster area with IR brightness temperature \( T_B \) less than \(-65^\circ\text{C}\), indicative of deep cumulus convection. Pertinent to our work is the occurrence of multiple bursts of convection, at intervals of approximately 24 h, prior to the tropical depression designation.

In order to simulate this multiple-burst effect, “pulses” of convective activity in the form of small-scale PV anomalies were added to the PV field at intervals (typically \( 0.5\tau_{e_dyy} \), half the initial eddy turnover time of the vortex, or about 1.5 days) during the time stepping process. A pulse consisted of a PV anomaly processing the same shape as the initial convective anomaly. Although the convective bursts shown in Fig. 16 are of extended duration, this feature is impractical to simulate in our model. Instead, our PV pulses occur all at once at the chosen time step. In the quasigeostrophic model, the wind and height fields adjust instantaneously to the PV field in accord with the invertibility relation [Eq. (5)]. Recall that gravity–inertia waves are excited in the balanced model. Model runs thus consisted of an initial vortex and asymmetric PV anomaly, which were pulsed with additional asymmetric PV anomalies each \( 0.5\tau_{e_dyy} \).

The pulse was applied just after the model fields were
written out. Since the runs were $2.5 \tau_{\text{eddy}}$ in duration, there were typically four pulses in addition to the initial anomaly. Because the shear time (see appendix A) $\tau_{\text{shcut}} = 0.2 \tau_{\text{eddy}}$, applying a pulse each $0.5 \tau_{\text{eddy}}$ just after the data was written out assured that the mean fields had stabilized subsequent to the pulse by the next output time.

Since the PV pulses had a $z$ dependence of $\cos \pi z$, as much negative as positive PV was added within the domain by each pulse, ensuring that the total PV of the fluid did not change. This is consistent with the general requirement noted in section 2b that, in the absence of friction and heat fluxes on boundaries, convection merely redistributes PV such that its mass-weighted integral is invariant.

In most of the simulations described here, the pulses grew in amplitude, with the amplitude $A_k$ of the $k$th pulse given by

$$A_k = (1 + e)^k A_0,$$

where $A_0$ is the initial pulse amplitude and $e = 0.2$. The amplitude of the convective maximum was increased to simulate the increasing relative vorticity being converged; a primitive equation effect not captured in the strict implementation of our quasigeostrophic model, which only converges planetary vorticity explicitly. The time rate of change of the potential vorticity is given by Eq. (7); thus the amplitude of the first pulse is given by $A_1 = (1 + e)A_0$, where $e$ represents the contribution of the relative vorticity; similarly $A_2 = (1 + e)A_1 = (1 + e)^2 A_0$, and so forth. As we will see below, if the pulse amplitude does not increase, considerably less spinup is achieved.

Figure 17 shows the azimuthal mean PV fields and azimuthal mean tangential winds at the lowest level as a function of time for a pulsed experiment with a two-cluster convective anomaly, with pulse amplitude increasing according to Eq. (15). The vortex tangential winds increase by 10 m s$^{-1}$, yielding a final maximum tangential wind of 16 m s$^{-1}$.

Figure 18 shows the azimuthal mean potential vorticity, flow potential temperature, flow geopotential, and tangential wind plotted as a function of $r$ and $z$ at $T = 2.5$ for the two-cluster pulsed asymmetry. Anticyclonic flow at upper levels is evident. Another feature of interest is the strong warm core ($\approx 5$ K) that forms at the center of the vortex (one nondimensional temperature unit corresponds to 30 K). In section 5c the formation of the warm core is discussed in greater detail.

For comparison, an experiment in which the pulse amplitude remained fixed with time [$e = 0$ in Eq. (15)] was also performed. Because the successive pulses of fixed PV tend to contribute a fixed value of PV to the vortex, in this case less spinup was obtained. Since the fluid is incompressible, particles can replace but not overlap each other. Thus as the high PV particles are attracted toward the center, they tend to pile up at the outside radius of the inner PV maximum, leading to a broadening of the inner PV maximum with time.

An experiment with single-cluster pulsed convective forcing was also performed. In this experiment a spinup of approximately 7.0 m s$^{-1}$ was obtained. As in the unpulsed axisymmetrization experiments, the single-cluster anomaly gives less spinup than the two-cluster anomaly. If the low-level spinup scaled linearly with the amount of positive PV injected, the single-cluster anomaly would be expected to contribute only 50% as much PV to the vortex as the double cluster does. Actually, it contributes considerably more than 50%; the single-cluster anomaly appears to be more efficient at the spinup process. This greater efficiency is likely a manifestation of the nonlinear feedback effect discussed in section 4f.

Plots of the time evolution of the mean and asymmetric PV fields on the highest and lowest levels for the pulsed single-cluster asymmetry show that at low to midlevels the positive anomaly is pulled into the main vortex during the symmetrization process. At high levels the negative anomaly is repelled from the main vortex; the anomaly rotates around the parent vortex and little or no axisymmetrization occurs.

Figure 19 shows $r-z$ mean fields for the pulsed experiment with the single-cluster convective anomaly. An upper-level anticyclone and warm core are evident in this figure.

b. Sensitivity tests

Recognizing the chaotic nature of cumulus convection, it is important to demonstrate that the hypothesis proposed here is not sensitive to details of how convection is represented in our model. Strictly speaking, this problem should be studied with a full physics model capable of representing cumulus convection and mesoscale dynamics. Such an analysis is beyond the scope of the present study. As a substitute, we have performed some simple sensitivity tests with the quasigeostrophic model, which assure that our results are essentially independent of details of the pulsing scheme such as the pulse frequency and location. A summary of the sensitivity tests is given in Table 4.

The first of these tests showed that our results are actually quite sensitive to the radial location of the convection. We performed a simulation that was the same as the nominal two-cluster experiment except that the two convective anomalies were placed at 400 km, rather than 250 km, from the center of the vortex. The pulse amplitude in this run was increased according to Eq. (15). For the 400-km case, the tangential wind maximum broadened relative to the 250-km case. The maximum spinup, 4.9 m s$^{-1}$, occurred at a radius of approximately 350 km. These results are qualitatively consistent with the results for the relaxation experiments described in section 4c.

To test for sensitivity to the pulse location when the
Fig. 17. Time evolution of the azimuthal-mean PV field (\(Q\)) and azimuthal-mean tangential wind (\(U\)) at \(z = 0\) for a pulsed two-cluster PV asymmetry. To obtain the tangential wind in m s\(^{-1}\), multiply the nondimensional velocity by 100. To obtain the radial displacement in km, multiply the nondimensional quantity by 1000.
pulse anomalies were at the same radius from the vortex but had different orientations, we created what we refer to as “scrambled” pulses. The initial two-cluster convective anomalies were located at 250 km along the x axis, as usual. However, the first secondary pulse anomalies, though also in the two-cluster configuration, were placed at 250 km along the y axis, rotated 90° from the initial anomaly. The second pulse was rotated back 90° to the position of the initial anomaly, and thereafter the pulses alternated between these two orientations along the x and y axes. The scrambled pulse runs gave quite similar results to the normal runs, with a maximum spinup of 10.3 m s⁻¹.

The dependence of the spinup on the pulse frequency was also tested. The vortex’s shear time is 5.6 × 10⁴ s, while the eddy turnover time τₑddy is 2.5 × 10⁵ s. Since τₛhear is considerably less than τₑddy and we normally pulse the system every 0.5τₑddy, the pulsed asymmetries tend to be sheared well before the onset of the next pulse. Thus, we expect that for a fixed total number of convective pulses, the mean tangential winds and thus the spinup should be approximately independent of the pulse frequency. To check this, we performed experiments for both the single- and the double-cluster cases with the pulse frequency doubled, but with the same total number of pulses as in the nominal runs. These experiments correspond to all of the spinup occurring in approximately 3.6 days rather than approximately 7.3 days. The values of maximum δu differ by about 1–2 m s⁻¹ from the lower-frequency values. Thus, as expected, the frequency of the convective pulsing in this frequency range does not significantly affect our results. In fact, if the convective anomalies were stronger, the spinup time could be considerably shorter. It should be recalled that the amplitudes used for the PV anomalies are quite conservative, given our estimate of tropical convective heating rates in section 2b. In the relaxation experiments, we find that axisymmetrization at middle
to low levels occurs even when the PV anomaly amplitude is increased by a factor of 3 over its nominal value. The maximum spin up in this experiment is 7 m s^{-1}, yielding a final $\delta \bar{v}_{\max}$ of 11.5 m s^{-1}.

c. Warm core formation

Figure 20 shows the time evolution of the azimuthal mean-potential temperature deviation from the resting atmosphere for the nominal two-cluster pulsed convective anomaly. As before, one nondimensional temperature unit corresponds to 30 K. The azimuthal mean vortex has an initial warm core of approximately 0.5 K at $z = 0.5$. The warm anomaly arises from the wave-number 0 component of the imposed asymmetry.

In the course of the experiment, which corresponds to $2.5 \tau_{\text{eddy}}$ or about 7.3 days, the warm core intensifies to a 5 K level. As demonstrated above, our model can actually achieve full spinup in $1.25 \tau_{\text{eddy}}$ or about 3.6 days. Thus we can demonstrate the buildup of a 5 K warm core on a reasonable timescale for tropical cyclogenesis. This is a significant result, as it links the formation of a warm core to the axisymmetrization process.
To investigate the mechanisms responsible for the warm core structure under multiple pulsing, we consider first the azimuthal mean-vertical velocity deduced from the thermodynamic Eq. (6). Normally, in multiple-pulsed runs the pulsing occurs each $0.5\tau_{\text{eddy}}$, just after the model fields and azimuthal averages are written out. Since the shear time $\tau_{\text{shear}} = 0.2\tau_{\text{eddy}}$, with the conventional pulsing scheme, the mean-field quantities have stabilized subsequent to the pulse before they are output. In order to focus on the relaxation process, we performed an additional pulsing experiment with the pulses occurring $0.5\tau_{\text{shear}}$ before each $0.5\tau_{\text{eddy}}$ output time. Thus, the outputs occurred during rather than after the symmetrization process.

Fig. 20. Time evolution of the azimuthal mean-potential temperature deviation from the resting atmosphere for the nominal pulsed two-cluster convective experiment. To obtain temperature in K, multiply the nondimensional quantity by 30. To obtain radial displacement in km, multiply the nondimensional quantity by 1000.
Figure 21 shows the mean azimuthal vertical velocity on z-surfaces $z = 0.75$, $0.5$, and $0.25$ at $T = 0.5$–$2.5$. One nondimensional unit of $\overline{w}$ corresponds to $1.0 \text{ m s}^{-1}$. For $T = 0.5$–$1.0$, most of the vertical velocities are of order $0.01$–$0.02 \text{ cm s}^{-1}$. These vertical velocities are too small to account for the observed temperature increase of the vortex by mean subsidence warming alone during this period. In $1.0T_{\text{eddy}}$, the expected temperature change of the vortex core due to mean subsidence is

$$\Delta \theta = \frac{\theta_0}{g} N^2 \tau_{\text{eddy}} = 0.075 - 0.15 \text{ K}.$$  

The observed warming in the first $T_{\text{eddy}}$ is about $2 \text{ K}$. By $T = 1.0$, however, the $z = 0.25$ level shows subsidence of nearly $0.1 \text{ cm s}^{-1}$. Vertical velocities at the other displayed levels remain small until $T = 2.0$, when we observe mean subsidence at all levels within the radius of maximum winds with maximum vertical velocities in the range $0.1$–$0.4 \text{ cm s}^{-1}$. Subsidence of order $0.1 \text{ cm s}^{-1}$ then persists at all levels in the inner core through $T = 2.5$. Thus, we expect subsidence warming to be more significant at later times.

6. Comparison with observations

We have examined the results of a published rawinsonde composite study of tropical cloud cluster evolution and cyclogenesis in the western North Pacific (Lee 1986, 1989a,b). In his study, Lee compared a composite of 341 selected developing cyclones to 332 “nonpersistent nongenesis” and 328 “persistent nongenesis” tropical cloud clusters. Clusters that could be observed only within one 24-h time period were defined as non-persistent nongenesis clusters, and those that were visible for two or more days were termed persistent nongenesis clusters. To study time evolution, composites for the developing cyclones were formed for various evolutionary time stages. The cyclone system was selected for inclusion in the composites when it was identified as a tropical depression. This was defined as evolutionary stage 3. Stage 3 included the first two 12-h time periods following identification as a tropical depression. Stage 4 included the two 12-h periods immediately after stage 3, stage 2 included the 24 h immediately before stage 3, and stage 1 included the 24 h prior to stage 2.

As a test of the relevance of our quasigeostrophic model, it is of interest to note the Rossby numbers at the radius of maximum winds corresponding to Lee’s Fig. 16, which shows a cross section of the azimuthal mean-tangential wind for the various stages of genesis. At genesis stage 1,

$$\text{Ro} = \frac{4 \text{ m s}^{-1}}{(2.8 \times 10^{-3} \text{ s}^{-1})(5.5 \times 10^4 \text{ m})} = 0.26 \ll 1;$$

for this stage of genesis the quasigeostrophic approximation is a good one. For genesis stage 5,

$$\text{Ro} = \frac{9 \text{ m s}^{-1}}{(3.5 \times 10^{-5} \text{ s}^{-1})(2 \times 10^4 \text{ m})} = 1.3;$$

at this stage the quasigeostrophic approximation is no longer strictly applicable.

The source of the increase in cyclonic circulation was investigated using a tangential-momentum budget analysis in a coordinate system moving with the storm system. Denoting the azimuthal mean-tangential wind by $\overline{\mathbf{v}}$, its depth-integrated time rate of change is given by

$$\int_{\text{troposphere}} \frac{\partial \overline{\mathbf{v}}}{\partial t} dp = \text{mean terms} + \text{eddy terms} + \text{motion terms} + \text{surface friction}.$$

The mean, motion, and surface friction terms can be determined from the composited data (see Lee’s papers for details). The eddy term is calculated as the residual of the other terms; it also includes any other residual effects such as data errors. It is important to note that any vertical fluxes of angular momentum are also included in Lee’s eddy term.

Lee’s tangential-momentum budget results are shown in Table 5. It is evident that for the genesis case the contributions from the eddy terms are quite large compared to all other terms, particularly at stages 1 and 2. The importance of the eddy terms in the early stages of cyclogenesis is consistent with our theory, which posits large inward fluxes of eddy angular momentum on synoptic scales ($1^\circ$–$5^\circ$ lat) subsequent to each convective burst. However, because Lee’s eddy term is determined by the residual method, Lee’s large eddy fluxes cannot be directly identified with our horizontal eddy fluxes; for example, vertical eddy fluxes could also contribute to Lee’s residual. The importance of eddy-momentum fluxes on the larger scale ($5^\circ$–$9^\circ$ lat) is not inconsistent with suggested environmental influences on tropical cyclogenesis (Challa and Pfeffer 1980, 1990; Pfeffer and Challa 1981; Montgomery and Farrell 1993).

7. Summary and conclusions

In this work we have examined the dynamics of the interaction of moist penetrative convection with a larger-scale vortex. Convection is parameterized by its estimated net effect on the potential vorticity field. Convection could be initiated by environmental asymmetries such as those described by Challa and Pfeffer (1980, 1990), Pfeffer and Challa (1981), Montgomery and Farrell (1993), and Challa et al. (1998), or by mesoscale processes. Here we provide an explicit explanation of how asymmetric convective-scale rotational kinetic energy is transformed to kinetic energy of the large-scale vortex.

For simplicity, a three-dimensional quasigeostrophic balance model is used to elucidate the underlying dynamics. This approach has the virtue of exciting gravity waves, allowing one to focus exclusively on the rota-
Fig. 21. Azimuthal-mean vertical velocity $\mathbf{v}$ at model levels $z = 0.75, 0.5,$ and 0.25 at $T = 0.5, 1.0, 1.5, 2.0, \text{ and } 2.5$ for a pulsed two-cluster experiment with pulses applied 0.5 $r_m$ before each output time. One nondimensional vertical velocity unit equals 1 m s$^{-1}$. One nondimensional unit of radial displacement equals 1000 km.
VORTEX ASYMMETRIZATION

Table 5. Lee’s tangential momentum budget analysis (from Lee 1989b). Units are m s^{-1} d^{-1}.

<table>
<thead>
<tr>
<th></th>
<th>Mean terms</th>
<th>Eddy terms</th>
<th>Motion terms</th>
<th>Surface friction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen-S1</td>
<td>1.2</td>
<td>+0.4</td>
<td>+0.4</td>
<td>+0.4</td>
</tr>
<tr>
<td>Gen-S2</td>
<td>0.0</td>
<td>+0.3</td>
<td>+1.2</td>
<td>+0.9</td>
</tr>
<tr>
<td>Gen-S3</td>
<td>0.2</td>
<td>+0.4</td>
<td>+1.4</td>
<td>+1.0</td>
</tr>
</tbody>
</table>

With reasonable assumptions about the magnitude of PV injection associated with moist penetrative convection, we obtain spinup to a 15 m s^{-1} cyclone on realistic timescales. Simulation of a midlevel vortex with peripheral penetrative convection shows that axisymmetrization results in the spinup of a surface cyclone. A warm-core vortex of order 5 K forms as a natural consequence of the axisymmetrization process. In the relaxation experiments, both eddy-heat and eddy-momentum fluxes are found to contribute to the development of the warm core. In the pulsed convective experiments, subsidence warming is initially too small to account for the observed warming; however, after the first \( t_{e_dy} \) subsidence warming is significant.

At the finite amplitudes determined consistent with our calculation of the magnitude of PV injection, the spinup obtained shows greater than linear dependence on the amplitude. This fact implies the existence of a nonlinear feedback mechanism associated with convectively forced vortex Rossby waves. Discrete neutral or weakly unstable vortex Rossby modes are observed to propagate azimuthally around the vortex center both at upper and lower levels; the possible role that these modes may play in orchestrating future convection, which may further intensify the vortex requires additional study.

Numerical simulations of tropical cyclogenesis using a full physics model (taking into account horizontal advection, vertical advection, stretching, tilting of horizontal vorticity, and friction) show that the dominant contributions to the low-level vertical vorticity tendency are horizontal advection and stretching (Kurihara and Tuleya 1981). This provides evidence that our idealized model is not too idealized, that is, that we have incorporated the fundamental processes producing the storm’s vertical vorticity even though we have neglected, for example, boundary layer friction and the complex mesoscale interactions leading to tilting of horizontal vorticity. Observations (Lee 1986, 1989a,b) suggest that small-scale eddy processes are important at the early stages of tropical cyclogenesis. These results are consistent with our expectations for vortex development by axisymmetrization in the presence of moist penetrative convection.

Further observations and carefully designed primitive equation modeling experiments are needed, however, to confirm or falsify the theory described here. These include observations of patches of low-level cyclonic relative vorticity in association with convection near the incipient vortex, such as a mesoscale convectively generated vortex. While one of the ultimate goals of an observational field program for tropical cyclogenesis should be to construct finescale PV maps, for convenience we suggest the use of the absolute vorticity as a useful proxy for PV. A key process in our theory is the shearing of such vorticity patches by the incipient vortex. Accompanying the shearing one would like to observe the upgradient transport of vorticity into the vortex and the downgradient transport into surrounding vorticity filaments. As the dynamics of vortex Rossby waves have been shown to usefully characterize the wave–mean flow interaction for the PV anomaly amplitudes considered here, it would also be of interest to observe wavelike features in the PV (or vorticity) field that are consistent with their local dispersion relation derived in Montgomery and Kallenbach (1997).
ciple, this dispersion relation can be used to distinguish vortex Rossby waves from gravity–inertia waves in both observational and modeling data.

Future work will test the theoretical conclusions developed here in a fully nonlinear three-dimensional primitive equation model and an asymmetric balance model, with moist processes represented in a more realistic manner.

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APPENDIX A

The Quasigeostrophic Model

The numerical solution of Eqs. (4)–(6) follows the finite difference method used by Montgomery and Farrell (1992) suitably modified to simulate quasigeostrophic dynamics. Time stepping is performed by a two-step predictor–corrector Adams–Bashforth method (Gazdag 1976). The advective terms of Eqs. (4) and (6) are calculated using fourth-order Arakawa advection (Arakawa 1966) with the sign correction as noted by Orszag (1971) in order to minimize dispersion errors. All other derivatives employ second-order centered differences.

The vertical grid spacing \( dz = 1.25 \text{ km} \), corresponding to nine vertical levels. The horizontal grid spacing \( (dx, dy) \) is typically \( 13.9 \text{ km} \), corresponding to about 14 grid points inside the radius of maximum winds of the initial circular basic-state vortex, and a total of \( 144 \times 144 \) total points in \( x \) and \( y \). The Courant–Friedrichs–Lewy (CFL) condition then requires a time step less than or equal to \( 2.78 \times 10^3 \text{ s} \). We generally used a time step with a factor of 4–8 smaller than the CFL limit. For the single cluster relaxation (section 4c) and midlevel vortex (section 4d) configurations we decreased the horizontal grid spacing to 7.5 km by increasing the number of \( (x, y, z) \) points in the model to \( 200 \times 200 \times 9 \) and shrinking the domain size from \( 2000 \times 2000 \times 10 \) to \( 1500 \times 1500 \times 10 \) km.

The model also includes second-order horizontal diffusion of \( Q \). Diffusion is added to remove small-scale PV associated with the potential enstrophy cascade. The value of the diffusion coefficient is based on two characteristic inviscid timescales, \( 1/\tau_{\text{eddy}} \) and \( \tau_{\text{shear}} \). The eddy turnover time of our barotropic vortex is initially \( 2.5 \times 10^3 \text{ s} \) or approximately 2.91 days. The characteristic shear time is calculated based on the inverse of the local radial shear in a circular vortex,

\[
\tau_{\text{shear}} = \left( \frac{d\tilde{\Omega}}{dr} \right)^{-1},
\]

where \( \tilde{\Omega} = \bar{\Omega}(r)/r \) is the angular velocity of the circular vortex in geostrophic and hydrostatic balance. For our vortex the maximum radial shear occurs initially near a nondimensional radius \( r = 0.15 \) and has a dimensional value of \( 0.18 \times 10^{-4} \text{ s}^{-1} \), resulting in a dimensional shear time \( \tau_{\text{shear}} = 5.6 \times 10^4 \text{ s} \). This yields a diffusivity based on the vortex shear of \( \nu_{\text{shear}} = (\text{horizontal grid spacing})^2/\tau_{\text{shear}} = 3.5 \times 10^3 \text{ m}^2 \text{ s}^{-1} \). Corresponding to \( 1/2\tau_{\text{eddy}} \), with \( \tau_{\text{eddy}} \) having a dimensional value of \( 2.5 \times 10^3 \text{ s} \), we find \( \nu_{\text{eddy}} = (\text{horizontal grid spacing})^2/(2\tau_{\text{eddy}}) = 1.5 \times 10^4 \text{ m}^2 \text{ s}^{-1} \). Based on these calculations and examination of the model results at small scales, we chose \( \nu = 4.2 \times 10^2 \text{ m}^2 \text{ s}^{-1} \), a factor of 6 smaller than the mean value of \( \nu_{\text{eddy}} \) and \( \nu_{\text{shear}} \). The results presented here have been verified to be insensitive to the precise value of \( \nu \).

In the absence of heating and friction the sum of the total kinetic energy \( K \) per unit mass plus the available potential energy \( P \) per unit mass integrated over the domain is conserved. The potential enstrophy is also conserved if the vertical boundaries are isothermal, as is the case for all initial conditions used here. We expect to see a slight decrease in energy and potential enstrophy with time due to the explicit diffusion of PV in the numerical model. Each of these quantities decreased by less than 0.5% in a 1.5 eddy-turnover-time run.

We tested our results for sensitivity to the time step size and the required convergence tolerance of the invertibility solver. No sensitivity to these parameters was found.

APPENDIX B

The Nondivergent Vorticity Equation Model

Although the nondivergent vorticity equation can be simulated with the quasigeostrophic model by simplifying to barotropic dynamics and interpreting \( \phi \) as the streamfunction, the relative contribution from the wave–mean flow and wave–wave interaction are assessed directly and naturally in the semispectral representation. In addition, comparatively higher temporal and spatial resolution can be obtained. The model employed extends the linear numerical model of Montgomery and Kallenbach (1997) to include the nonlinear advective terms. In the nondivergent model, the perturbation streamfunction \( \psi' \) and the perturbation relative vorticity \( \zeta' \) are represented semispectrally:
\[ \psi'(r, \lambda, t) = \sum_{i=-N}^{N} \hat{\psi}_i(r, t)e^{i\lambda i} \]  

(\text{B.1})

and

\[ \zeta'(r, \lambda, t) = \sum_{m=-N}^{N} \hat{\zeta}_m(r, t)e^{im\lambda}, \]  

(\text{B.2})

where \( \psi_i \) and \( \zeta_m \) denote the azimuthal Fourier amplitude for streamfunction and vorticity, respectively, and an azimuthal wavenumber truncation of \( N = 8 \) is used for the example considered. The dimensional prognostic equation for the streamfunction is

\[ \frac{\partial \hat{\psi}_n}{\partial t} = \nabla^2 \hat{\psi}_n(r, t), \]  

(\text{B.3})

where

\[ \hat{F}_n(r, t) = \frac{1}{r} \sum_{m \neq n} \left[ ik \frac{\partial}{\partial r} (\hat{\psi}_m \hat{\zeta}_{n-k}) - in \hat{\zeta}_m \frac{\partial \hat{\psi}_n}{\partial r} \right] \]

\[ + i\nu \frac{d^2 \hat{\psi}_n}{dr^2} - i\nu \frac{d^2 \hat{\zeta}_n}{dr^2} + \nu \nabla^2 \hat{\psi}_n, \]  

(\text{B.4})

and \( n \neq k \) and \( \nabla^2 \hat{\psi}_n = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \hat{\psi}_n}{\partial r} \right) + \frac{\partial^2 \hat{\psi}_n}{\partial z^2} - \frac{n^2}{r^2} \). A diffusion coefficient \( \nu \) of 20 m^2 s^{-1} is used for this simulation. The inversion is carried out using a standard tri-diagonal solver. Radial derivatives are computed with centered second-order differences. Time stepping is performed using a fourth-order Runge-Kutta scheme (Abramowitz and Stegun 1972). The radial grid spacing is 8.5 km, with 200 radial points. Plots and results given in this paper have been nondimensionalized as for the quasigeostrophic model.

**APPENDIX C**

**Lagrangian Trajectories**

To further elucidate the lateral mixing processes taking place within the vortex, we track Lagrangian trajectories of PV particles, small elements of fluid that have a fixed value of PV and are advected by the geostrophic wind.\(^3\) The forward tracking algorithm finds the end location of a particle with a given initial location at \( t = 0 \) after a given amount of time (typically the model run time) has passed. The backward tracking algorithm finds, for a particle at a given location after the model has run, the location from which the particle originated at \( t = 0 \).

To show how the trajectory algorithms are implemented, we first discuss forward tracking. We consider the initial-value problem

\[ \frac{dx}{dt} = u(x_i, y_i, t) \]  

(\text{C.1})

\[ \frac{dy}{dt} = v(x_i, y_i, t) \]  

(\text{C.2})

for the \( i \)th PV particle, where \( u \) and \( v \) are the \( x \) and \( y \) velocity fields calculated by the model, and \( x_i(0) = \alpha \) and \( y_i(0) = \beta \) are given. Thus, for each of the \( i \) PV particles we have a pair of coupled differential equations, which we wish to solve for \( x_i(t) \) and \( y_i(t) \).

The equations are solved by fourth-order Runge-Kutta integration. To calculate the right-hand sides of Eqs. (C1) and (C2), gridded velocity fields at a chosen \( z \) level are output from the quasigeostrophic model at each time step. The values of \( u(x_i, y_i, t) \) and \( v(x_i, y_i, t) \) are obtained from the gridded velocities by bilinear interpolation.

Backward trajectories are only slightly more complicated. The Runge-Kutta algorithm must step through the velocity fields from the end (typically, \( t = 2\pi r_E \)) of the file to its beginning at \( t = 0 \). Because the trajectories are time-reversed (analogous to running a movie backward) the velocity vectors must also be reversed.

**REFERENCES**


