Chap. 8 Laminar flow

前面各章中，我們針對流體的特性已有了詳細的介紹。在這一章中我們將針對在大氣中會見到的 **Laminar flow** 做討論。

在前面幾章裡我們已經介紹過了 **Turbulent flow**，而在大氣之中，**Laminar flow** 和 **Turbulent flow** 時常相伴隨產生，但兩者特性卻完全不同。現在我們就以 **Turbulent flow** 做對照，來區分它和 **Laminar flow** 之相異性。

- **Laminar flow**：fluid moves in layers
- **Turbulent flow**：irregular motion，macroscopic mixing motion（the fluid mixes violently as moving along）

其實我們從上一章中所介紹的 Reynold number 也可以清楚的分別出 **Turbulent flow** 和 **Laminar flow**。

$$\mathrm{Re} = \frac{\mathrm{UL}}{\upsilon} < 3000 \Rightarrow \textbf{Laminar flow}$$

$\frac{U^2}{L}$ ：momentum

$\upsilon\nabla^2 u$ ：viscous force

若 Re 小：viscous force constraints the motion of fluid in parallel layers

$\Rightarrow$ **Laminar flow**

註：Re 越大，越有可能形成亂流，但並不是絕對;在不同的環境中，Re 有不同的臨界值。

下面我們再舉幾個例子，試著從數學的觀點來探討有關 **Laminar flow** 的問題。

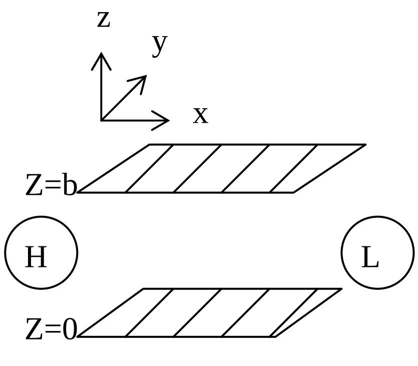
【e.g.】Laminar flow of incompressible flow through straight channels with parallel boundaries

u≠0，v＝0，w＝0

$$\Rightarrow \nabla\cdot\vec{u} = \frac{\partial u}{\partial x} = 0 \text{，u = u(t，y，z)}$$

$$\rho\frac{d\vec{u}}{dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{u}$$

x-component：



《圖 8-1》

$$\rho\left(\frac{\partial u}{\partial t}+u\cancel{\frac{\partial u}{\partial x}}+v\cancel{\frac{\partial u}{\partial y}}+w\cancel{\frac{\partial u}{\partial z}}\right)$$

$$=\rho g_x-\frac{\partial p}{\partial x}+\mu\left[\left(\cancel{\frac{\partial^2}{\partial x^2}}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u\right]$$

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u$$

$\Rightarrow \quad 0=-\dfrac{\partial p}{\partial y} \quad \Rightarrow$ p is indep. of y

$0=\rho g+\dfrac{\partial p}{\partial z}$ （靜力平衡）

if the fluid is steady laminar flow：

u = u (y , z) ⇒ u = u (z) (why ?)

$-\dfrac{\partial p}{\partial x}+\mu\dfrac{\partial^2 u}{\partial z^2}=0$ ，$\dfrac{\partial p}{\partial x}$ is indep. of z

B.C.：1. at z=0，u=0

2. at z=b，u=0

$$\int\left[-\frac{\partial p}{\partial x}+\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)=0\right]dz\Rightarrow\frac{-\partial p}{\partial x}z+\mu\frac{\partial u}{\partial z}=c_1\ ，$$

$$\int\left(\frac{-\partial p}{\partial x}z+\mu\frac{\partial u}{\partial z}=c_1\right)dz\Rightarrow-\frac{1}{2}\frac{\partial p}{\partial x}z^2+\mu u=c_1 z+c_2$$

$$\Rightarrow u=\frac{1}{2\mu}\frac{\partial p}{\partial x}z^2+C'_1z+C'_2 \qquad (8\text{-}1)$$

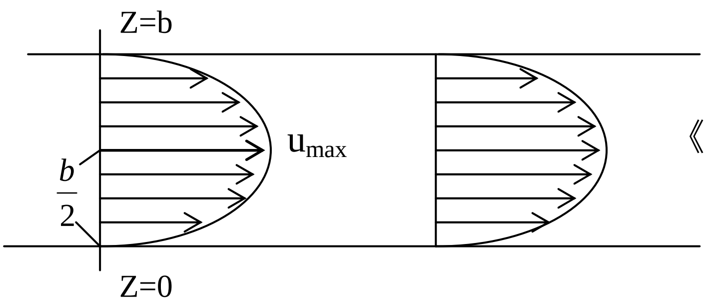
$\Rightarrow$ by B.C. 1， $0=C'_2$

by B.C. 2， $0=\dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}b^2+C'_1 b$

$\Rightarrow -\dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}b=C'_1$

$$\Rightarrow \varphi=\frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left(bz-z^2\right)=\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(z-\frac{b}{2}\right)^2-\frac{b^2}{4}\right]$$

$$\Rightarrow z=\frac{b}{2}\ ，u=u_{max}=\frac{1}{2\mu}\left(-\frac{\partial p}{\partial x}\right)\left[\left(z-\frac{b}{2}\right)^2-\frac{b^2}{4}\right]$$

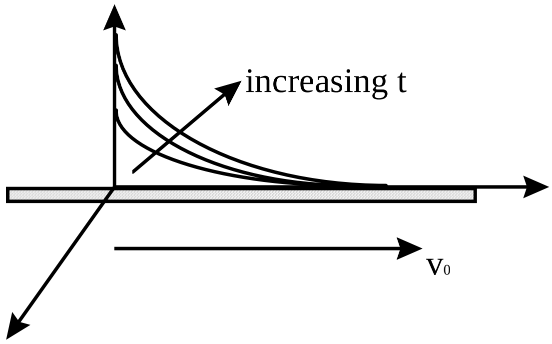


《圖 8-2》

現在我們再來看另一個例子。

【e.g.】

如《圖 8-3》；在一流體的底部有一平行板，以v_0的速度沿Y軸移動。



《圖 8-3》

$v \neq 0$ ， $\frac{\partial v}{\partial y} = 0$ (沿 y 軸各點相同)

$u=0$，$v=v(z，t)$，$w=0$

$$\Rightarrow \rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial z^2}$$

$$\Rightarrow \frac{\partial v}{\partial t} = \upsilon \frac{\partial^2 v}{\partial z^2} \qquad (8\text{-}2)$$

I.C. V＝0 for all z，at t≦0

B.C. 1. at z＝0，v＝v_0，t>0

   2. at z＝∞，v＝0，t＞0

There are at least three ways to solve this equation：

1. Separation of variables
2. Method of Laplace transform
3. Similarity solution

We use similarity solution to solve the equation：

z， t，ν ⇒ { 3 variables; 2 dimensions }

⇒3－2＝1，⇒non-dimensional parameter

$\pi = z^{a_{11}} t^{a_{12}} \upsilon^{a_{13}}$ ⇒a_{11}：a_{12}：a_{13}＝1：-1/2：-1/2

$$\Rightarrow \pi = z \Big/ \sqrt{\upsilon t}$$

令 $\eta = z \Big/ \sqrt{4\upsilon t}$ ，設$v=v_0 f(\eta)$

$$\frac{\partial v}{\partial t} = v_0 \frac{\partial f(\eta)}{\partial t} = v_0 \frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial t} = v_0 \frac{\partial f(\eta)}{\partial \eta}\left(-\frac{1}{2}\frac{\eta}{t}\right)$$

$$\frac{\partial^2 v}{\partial z^2} = v_0 \frac{\partial^2 f(\eta)}{\partial z^2} = v_0 \frac{\partial}{\partial z}\left(\frac{\partial f(\eta)}{\partial z}\right) = v_0 \frac{\partial}{\partial z}\left(\frac{\partial f(\eta)}{\partial \eta}\frac{\partial \eta}{\partial z}\right) \quad \rightarrow \frac{1}{\sqrt{4\nu t}}$$

$$= \frac{v_0}{\sqrt{4\upsilon t}}\frac{\partial^2 f(\eta)}{\partial \eta^2}\frac{\partial \eta}{\partial z} = \frac{v_0}{4\upsilon t}\frac{\partial^2 f(\eta)}{\partial \eta^2} = v_0 \frac{\eta^2}{z^2}\frac{\partial^2 f(\eta)}{\partial \eta^2} \qquad (8\text{-}3)$$

我們將式(8-3)代回式(8-2)中：

$$\Rightarrow -\frac{1}{2}\frac{\eta}{t}v_0\frac{df(\eta)}{d\eta} = \upsilon v_0\frac{\eta^2}{z^2}\frac{df^2(\eta)}{d\eta^2}$$

$$\frac{\partial^2 f(\eta)}{\partial\eta^2} + \left(\frac{z^2}{\upsilon 2t\eta^2}\right)\left(\eta\frac{df(\eta)}{d\eta}\right) = 0$$

$$\Rightarrow \begin{cases} \dfrac{d^2 f(\eta)}{d\eta^2} + 2\eta\dfrac{df(\eta)}{d\eta} = 0 \\ \text{B.C. 1.} \quad \eta = 0, \ V = V_0 \Rightarrow f(\eta) = 1 \\ \quad\quad\ \ 2. \quad \eta\to\infty, \ V\to 0 \Rightarrow f(\eta)\to 0 \end{cases}$$

$$\Rightarrow \frac{df(\eta)}{d\eta} = C_1 e^{-\eta^2} \Rightarrow f(\eta) = C_1\int_0^{\eta} e^{-\eta^2}d\eta + C_2 \leftarrow by\ B.C.$$

by B.C. $\begin{cases} 1. \quad \eta = 0, \ f(\eta) = 1 \Rightarrow C_2=1 \\ 2. \quad \eta\to\infty, \ f(\eta)\to 0 \Rightarrow C_1 = \dfrac{-1}{\int_0^{\infty} e^{-\eta^2}d\eta} = \dfrac{-2}{\sqrt{\pi}} \end{cases}$

Error function

$$f(\eta) = 1 - \frac{\int_0^{\eta} e^{-\eta^2}d\eta}{\int_0^{\infty} e^{-\eta^2}d\eta} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}d\eta = 1 - erf(\eta)$$

$$v = v_0 f(\eta) = v_0\left[1 - erf(\eta)\right] = v_0\left[1 - erf\left(\frac{z}{\sqrt{4\upsilon t}}\right)\right]$$

$$= v_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{\sqrt{4\upsilon t}}} e^{-\eta^2}d\eta\right)$$

現在我們舉一些例子來代入：

1. $\frac{V}{V_0} = 0.05 \Rightarrow \eta = 1.38$

$\Rightarrow z = 2.76\sqrt{\upsilon t}$

2. $\frac{V}{V_0} = 0.8 \Rightarrow \eta = 0.18$

$\Rightarrow z = 0.36\sqrt{\upsilon t}$

由上面這一個例子中，我們能夠了解下列幾點：

1. 隨著時間的增加，流體因邊界之黏滯效應所產生的速度變化範圍也會增加。
2. 若我們想知道在某一時刻高度 Z 之流體速度，我們只需找出 η 值即可導出其關係式!!

想一想，地球自轉，經過漫長時間後，大氣如何運轉？