Chap. 7 Dynamical Similarity

§7.1　Dynamically similar

空間中存在任意兩流體，兩者在 L（length scale）、V（velocity scale）等物理量上雖然有明顯的不同，但如果兩者滿足 mathematically identical　system：

1.在邊界具有幾何相似性

2.在每一點都具有相同的無因次參數控制著流體

則稱它們具有**動力相似性（dynamical similarity）**。

動力相似性是研究流體力學的一個重要方法；我們藉由因次的分析，將方程式中有因次的變數組合形成無因次參數，以減少問題中的變數，降低流動問題的複雜性；再者，利用兩流體間的動力相似性，我們也可由其中一已知的流體資料來推測另一流體的運動情形。因此，動力相似和因次分析對於流體力學的探討和實驗工作有相當助益。例如：風洞實驗、台電大樓模型實驗。

接下來，我們便要介紹如何求得無因次的參數。其中一種方法，便是直接由已知的關係式中去推算，以 Navier-Stokes eq.為例

$$\frac{d\bar{u}}{dt} = -\frac{1}{\rho}\nabla P + \bar{g} + \nu\nabla^2\bar{u} \qquad (7\text{-}1)$$

其中定義

length：L（m）$\xrightarrow{\text{綜觀尺度}}$ 10^6 m

velocity：U（m/s）

pressure：P（Pa）

time：L/U（s）$\xrightarrow{\text{綜觀尺度}}$ 10^5 s

並且定義無因次的變數

$$x^* = \frac{x}{L} \quad , \quad y^* = \frac{y}{L} \quad , \quad z^* = \frac{z}{L}$$

$$\bar{u}^* = \frac{u}{U} \quad , \quad p^* = \frac{p}{P} \quad , \quad t^* = \frac{Ut}{L}$$

$$(\because \nabla = \frac{\partial}{\partial x} = \frac{\partial}{\partial L x^*})$$

$$\nabla^* = \frac{\partial}{\partial x^*} = \frac{\partial}{\frac{\partial x}{L}} = L\frac{\partial}{\partial x} = L\nabla$$

（7-1）式則可寫成

$$\frac{L}{U^2}\left[\frac{dU\bar{u}^*}{d\left(\frac{L}{U}t^*\right)}=-\frac{1}{\rho}\frac{1}{L}\nabla^*\left(P\cdot p^*\right)+\bar{g}+\nu\frac{1}{L^2}\nabla^{*2}\left(U\bar{u}^*\right)\right] \qquad (7\text{-}2)$$

$$\Rightarrow \frac{d\bar{u}^*}{dt^*}=\frac{-P}{\rho U^2}\nabla^* p^*+\frac{Lg}{U^2}\frac{\bar{g}}{g}+\frac{\nu}{UL}\nabla^{*2}\bar{u}^*$$

如此一來，Navier-Stokes eq.便可完全由無因次項及無因次參數來表示；另外我們也對其中的無因次參數做以下的定義

$$Euler\ \#\equiv E=\frac{\rho U^2}{P}=\frac{\text{加速項}(inertial\ term)}{\text{氣壓梯度力}(pressure\ gradient\ force)}$$

$$Froude\ \#\equiv Fr=\frac{U}{\sqrt{gL}}=\left(\frac{\text{加速項}}{\text{重力項}}\right)^{\frac{1}{2}}$$

$$\mathrm{Re}\,ynold\ \#\equiv \mathrm{Re}=\frac{UL}{\nu}=\frac{\text{加速項}(inertial\ term)}{\text{摩擦項}(frictional\ term)}$$

又

$$(1)\ \frac{-\frac{1}{\rho}\nabla P}{\frac{d\vec{u}}{dt}}=\frac{-\frac{P}{\rho L}}{\frac{U^2}{L}}\frac{\nabla^* p*}{\frac{d\overrightarrow{u^*}}{dt^*}}=\frac{P}{\rho U^2}\frac{\nabla^* p^*}{\frac{d\overrightarrow{u^*}}{dt^*}}=\frac{1}{E}\frac{\nabla^* p^*}{\frac{d\overrightarrow{u^*}}{dt^*}}$$

$$(2)\ \frac{\vec{g}}{\frac{d\vec{u}}{dt}}=Fr^2\frac{\frac{\vec{g}}{g}}{\frac{d\overrightarrow{u^*}}{dt^*}}$$

$$(3)\ \frac{\nu\nabla^2\vec{u}}{\frac{d\vec{u}}{dt}}=\mathrm{Re}\frac{\nabla^{*^2}\overrightarrow{u^*}}{\frac{d\overrightarrow{u^*}}{dt^*}}$$

則（7-2）式便可以寫為

$$\frac{d\bar{u}^*}{dt^*}=-\frac{1}{E}\nabla^* p^*+\frac{1}{Fr^2}\frac{\bar{g}}{g}+\frac{1}{\mathrm{Re}}\bar{\nabla}^{*2}\bar{u}^*$$

§7.2　Buckingham　π -method

除了上一節所介紹的直接由方程式去推得無因次參數外，另一個常用的方法便是 **Buckingham　π -method**。Buckingham　π -method 敘

述的是在一個因次齊次(dimensionally homogeneous) 的方程式中，若有 m 個變數（$B_1, B_2, B_3 \cdots, B_m$）使得 $F(B_1, B_2 \cdots B_m) = C$（常數），並且可由 k 個基本因次（如 M，L，T 等）將此方程式完全描述；則我們可以將其轉換成（$m-k$）個 π -項，每一個 π -項都是無因次參數，而使得 $F(\pi_1, \pi_2 \cdots \pi_{m-k}) = C'$。其中

$$\pi_1 = B_1^{a_{11}} \cdot B_2^{a_{12}} \cdots\cdots\cdots B_k^{a_{1k}} \cdot B_{k+1}$$

$$\pi_2 = B_1^{a_{21}} \cdot B_2^{a_{22}} \cdots\cdots\cdots B_k^{a_{2k}} \cdot B_{k+2}$$

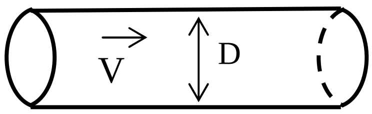
$$\vdots \qquad \vdots \qquad \vdots$$

$$\pi_{m-k} = B_1^{a_{(m-k)1}} \cdot B_2^{a_{(m-k)2}} \cdots B_k^{a_{(m-k)k}} \cdot B_m$$

條件：

1. Contained all the k- dimensions among themselves
2. Themselves do not form a dimensionless parameter

【e.g.】There is a steady flow of a Newtonian liquid through a long、straight、horizontal pipe of circular cross section，how does pressure change？



Sol：

已知此問題的主要相關變數有七個，即

ΔP：氣壓改變量 (pressure drop)
V：流體速度 (average velocity V of the pipe flow)
L：管長 (pipe length)
D：水管直徑 (pipe diameter)
ρ：流體密度 (fluid density)
μ：動力摩擦係數(fluid viscosity)
e：粗糙度 (average pipe wall roughness)

$$F(\Delta P, V, L, D, \rho, \mu, e) = C$$

$$m = 7$$

而基本因次爲L, M, T,即

$\Delta P = ML^{-1}T^{-2}$, $V = LT^{-1}$

$L = L$, $D = L$

$\rho = ML^{-3}$, $\mu = ML^{-1}T^{-1}$

e(粗糙度roughness length) $= L$

$$\therefore k = 3 \Rightarrow m - k = 7 - 3 = 4$$

由 Buckingham π -method

$$\Rightarrow \pi_1 = V^{a_{11}} \cdot D^{a_{12}} \cdot \rho^{a_{13}} \cdot \Delta P$$

$$\pi_2 = V^{a_{21}} \cdot D^{a_{22}} \cdot \rho^{a_{23}} \cdot L$$

$$\pi_3 = V^{a_{31}} \cdot D^{a_{32}} \cdot \rho^{a_{33}} \cdot \mu$$

$$\pi_4 = V^{a_{41}} \cdot D^{a_{42}} \cdot \rho^{a_{43}} \cdot e$$

因為π_1、π_2、π_3和π_4均為無因次，可以解得

$$\pi_1 = M^0 L^0 T^0 = (LT^{-1})^{a_{11}} \cdot L^{a_{12}} \cdot (ML^{-3})^{a_{13}} \cdot (ML^{-1}T^{-2}) = M^{a_{13}+1} \cdot L^{a_{11}+a_{12}-3a_{13}-1} \cdot T^{-a_{11}-}$$

$$\Rightarrow a_{11} = -2 \quad a_{12} = -1 \quad a_{13} = 0$$

$$\therefore \pi_1 = V^{-2} \cdot D^0 \cdot \rho^{-1} \cdot \Delta P = \frac{P}{V^2 \rho} = \frac{1}{E}$$

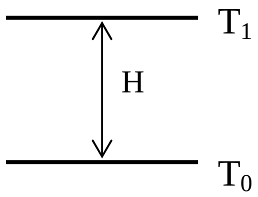
$$\left(\begin{aligned} &\pi_2 = V^0 \cdot D^{-1} \cdot \rho^0 \cdot L = \frac{L}{D} \\ &\pi_3 = \frac{\mu}{VD\rho} = \frac{\nu}{VD} = \frac{1}{\mathrm{Re}} \\ &\pi_4 = \frac{e}{D} \end{aligned} \right)$$

$\longleftarrow$請同學練習找出π_2、π_3和π_4這三個無因次參數

$$F(\pi_1, \pi_2, \pi_3, \pi_4) = C'$$

$$F\left(E, \frac{L}{D}, \mathrm{Re}, \frac{e}{D}\right) = C''$$

Application： Raleigh-Bernard　Convection



有六個變數：T_0、 T_1 、g、H 、ν 、κ
有三個因次：temp.、 length 、time

$\Longrightarrow$需三個無因次參數

$$\Rightarrow \begin{cases} Ra = g\left(\dfrac{T_1 - T_0}{T_0}\right)\dfrac{H^3}{\nu\kappa} \\ N = \dfrac{T_1}{T_2} \\ \sigma = \dfrac{\nu}{\kappa} \end{cases}$$

(Ra # 超過某一臨界值，就有機會產生對流)

§7.3　Significance of common non-dimensional parameters

前面介紹過如何求無因次參數的方法後，這一節便要針對一些常用的無因次參數做一簡單說明。

1.**Reynold #**

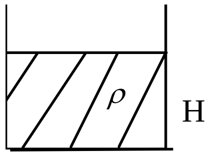
決定流體亂流的程度 (the ratio of inertial force to viscous force)

$$\mathrm{Re} = \frac{UL}{\nu} = \frac{inertial\ \ force}{viscous\ \ force}$$

2.**Froude #**

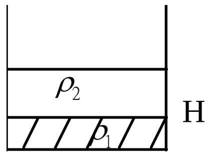
$$Fr = \frac{U}{gL} = \left(\frac{inertial\ \ force}{gravity\ \ force}\right)^{1/2}$$

(1) shallow-water wave



$\rho = constant$, $c = \sqrt{gH}$

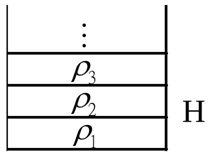
(2) internal gravity wave



$$g' = g\frac{\rho_1 - \rho_2}{\rho_1} \rightarrow reduced\ \ gravity$$

$$c = \sqrt{g'H} \quad , \quad Fr = \frac{U}{\sqrt{g'l}}$$

(3) for a continuous stratified fluid



$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z} = \frac{-g}{\rho_0}\frac{d\rho_0}{dz} \quad (N \equiv buoyancy\ \ frequence)$$

$$\Rightarrow Fr = \frac{U}{N \cdot H}$$

3.**Richardson #**

$$Ri = \frac{g'l}{U^2} \quad (\text{兩層})$$

$$= \frac{N^2 l^2}{U^2} = \frac{N^2}{\left(\frac{dU}{dz}\right)^2} \quad (\text{多層})$$

Ri<0.25→instability

4.**Mach #**

$$M = \frac{U}{C} = \left(\frac{inertial\ \ force}{compressible\ \ force} \right)^{\frac{1}{2}} = \left(\frac{\rho U^2 / l}{\rho C^2 / l} \right)^{\frac{1}{2}}$$

M<1→subsonic

M>1→supersonic

5.**Prandtl #**

$$\Pr = \frac{kinematic\ \ vis\cos ity}{thermal\ \ diffusivity} = \frac{\nu}{\kappa} = \frac{\mu / \rho}{k / \rho C_p} = \frac{C_p \mu}{k}$$

6.**Rossby #**

$$R_0 = \frac{U}{fL}$$ **其中** f(地轉風) $= 2\Omega \sin \Phi$

→ Ω：地球每秒轉動弳度:$7.292 \times 10^{-5}\, s^{-1}$