Chap 10 Waves

§10.1 Properties of Waves

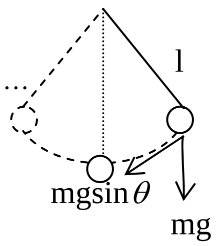
『波動』也許是所有物理現象最基本的表現形式。在這一節中，我們將藉由振動的機制來說明一些波動的基本性質。有關波動在大氣中的實例，將在大氣動力學（下）做更詳細的描述。

e.g. Linear harmonic oscillation

$\longrightarrow$ Simple pendulum

$$m\frac{d^2(l\theta)}{dt^2} = -mg\sin\theta$$

$\nu t = \pi$ 、3π ......



$\nu t = 0$ 、2π ......

《圖 10-1》

Assume: θ is small

then

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...... \sim \theta$$

$$\longrightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$Def.$ $\nu = \sqrt{\frac{g}{l}}$ (angular frequency)

$$\Rightarrow \frac{d^2\theta}{dt^2} + \nu^2\theta = 0$$

可解得

$\theta = 6_{N}$

$$\theta = \theta_1\cos\nu t + \theta_2\sin\nu t$$

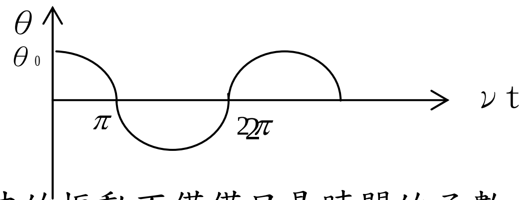
$$= \theta_0\cos(\nu t - \alpha)$$

(α → phase, θ_0 → amplitude)

$$= \theta_0\cos[\nu(t + \frac{2n\pi}{\alpha}) - \alpha] \quad , n \in I$$

$$T = \frac{2\pi}{\nu} = 2\pi\sqrt{\frac{l}{g}}$$

$$I.C. \qquad t=0, \theta=\theta_0 \Rightarrow \alpha=0 \Rightarrow \theta=\theta_0\cos(\nu t)$$



《圖 10-2》

由於波的振動不僅僅只是時間的函數，所以接下來我們將加入波在空間中 propagation 的概念。

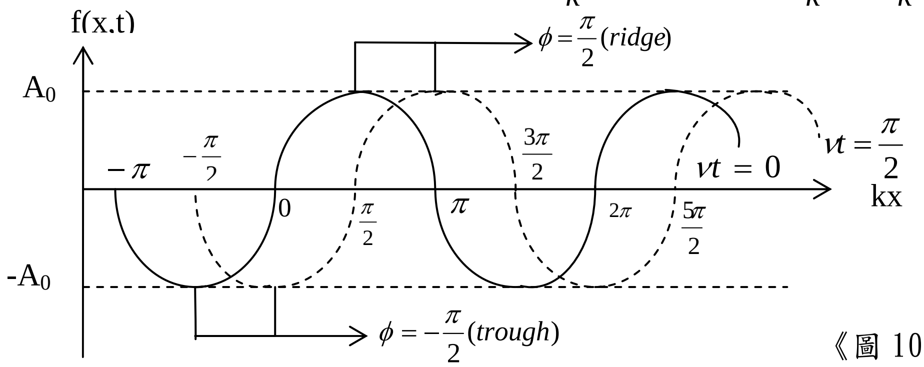
· **Propagating waves:**

Phase depends not only on time，but also on one or more space variable(s).

e.g. 1-Dim

$$\phi = kx - \nu t - \alpha$$

$$f(x,t) = A_0 \sin(kx-\nu t) = A_0 \sin k(x-\frac{\nu}{k}t) = A_0 \sin k(x - \frac{\nu}{k}t + \frac{2m\pi}{k})$$



《圖 10-3》

$\because kx = 2\pi$ 移動 1/4 波長

$$\left.\begin{array}{l}\Delta x = \dfrac{\pi}{2k} \\ \Delta t = \dfrac{\pi}{2\nu}\end{array}\right\} \Rightarrow c = \frac{\Delta x}{\Delta t} = \frac{\nu}{k}$$

Def. 波長 L= $\frac{2\pi}{k}$

$$k = \frac{2\pi}{L} = wave\ \ number$$

· Following any specific phase (**e.g.** ridge， $\phi = \frac{\pi}{2}$)：

$$0 = k(\frac{dx}{dt})_\phi - \nu$$

phase　speed：

$$c = (\frac{dx}{dt})_\phi = \frac{\nu}{k}$$

其中

$\nu = (-\frac{\partial\phi}{\partial t})_x =$ number of crests passing per unit time

$k = (\frac{\partial\phi}{\partial x})_t =$ number of spacial undulation in a unit distance

$$c = (\frac{dx}{dt})_\phi = \frac{-\frac{\partial\phi}{\partial t}}{\frac{\partial\phi}{\partial x}} = \frac{\nu}{k}$$

· **1-Dim**

$$f(x,t) = A_0 \sin(kx - \nu t)$$

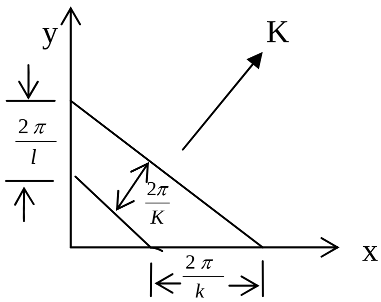
· **3-Dim**

Assume $f(x,y,z,t) = A_0 \sin(kx + ly + mz - \nu t)$

$$= A_0 \sin(\overrightarrow{K} \cdot \overrightarrow{X} - \nu t)$$

$$\Rightarrow \quad c = \frac{\nu}{k} \quad , \quad k = \frac{2\pi}{L}$$

振幅=const.→plane wave



《圖 10-4》

其中　$\overrightarrow{K} = (k, l, m)$ (稱為 wave number vector)

$\overrightarrow{X} = (x, y, z)$

$$K^2 = k^2 + l^2 + m^2$$

$$c = \frac{\nu}{K} \Rightarrow \vec{c} = \frac{\nu}{K}\frac{\overrightarrow{K}}{K} \Rightarrow c_x = \frac{\nu}{k} \;,\; c_y = \frac{\nu}{l} \;,\; c_z = \frac{\nu}{m}$$

Note：

$$c^2 \neq {c_x}^2 + {c_y}^2 + {c_z}^2$$

§10.2 Dispersive waves and Group velocity

・**Dispersive waves**：

1-D

$c = \frac{\nu}{k} \overset{\text{if}}{=\!=} f(k) \Rightarrow$ dispersive waves (頻散波)

$\neq f(k) \Rightarrow$ non-dispersive waves(非頻散波)

Note：

The shape of the dispersive wave pattern changes in time.

・**Group velocity**：

Assume two waves of slightly different wave numbers and frequency：

Wave 1 wave number：(k+Δk)

frequency：$(\nu + \Delta\nu)$

$$\Rightarrow f_1(x,t) = A_0 e^{i[(k+\Delta k)x - (\nu+\Delta\nu)t]} \Rightarrow c_1 = \frac{\nu + \Delta\nu}{k + \Delta k}$$

Wave 2 wave number：(k−Δk)

frequency：$(\nu - \Delta\nu)$

$$\Rightarrow f_2(x,t) = A_0 e^{i[(k-\Delta k)x - (\nu-\Delta\nu)t]} \Rightarrow c_2 = \frac{\nu - \Delta\nu}{k - \Delta k}$$

(Mathematical rules：$e^{i\phi} = \cos\phi + i\sin\phi$，

$\cos\phi = \mathrm{Re}[e^{i\phi}]$ ， $\sin\phi = \mathrm{Im}[e^{i\phi}]$)

$$\xrightarrow{\text{兩個波疊加}}$$

$$f(x,t) = f_1(x,t) + f_2(x,t) = A_0\left[e^{i(kx-\nu t)}\right]\left[e^{i[(\Delta k)x - \Delta\nu t]} + e^{-i[(\Delta k)x - \Delta\nu t]}\right]$$

$$= \underbrace{2A_0 \cos[(\Delta k)x - (\Delta\nu)t]}_{(1)} \underbrace{e^{i(kx-\nu t)}}_{(2)}$$

物理意義：

(1) Group velocity：the speed for amplitude propagating

[振幅(即能量)的傳遞速度⇒振幅本身也會振動]

Def.

$$c_g = \frac{\Delta\nu}{\Delta k} \xrightarrow{\lim} \frac{\partial\nu}{\partial k} = \frac{\partial(c \cdot k)}{\partial k}$$

$$C_g \xlongequal{\text{For non-dispersive wave}} c\,\frac{\partial k}{\partial k} = c$$

(2)相位速度：$c = \dfrac{\nu}{k} = \dfrac{\text{wave1 與 wave2 之 } k \text{ 的平均值}}{\text{wave1 與 wave2 之 } \nu \text{ 的平均值}}$

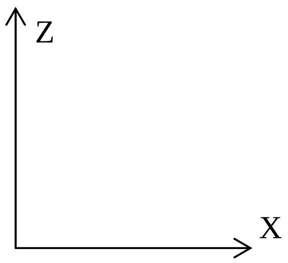
Examples：

(1) Sound waves

$$c = (\gamma RT)^{\frac{1}{2}} \quad , \quad \gamma = \frac{c_p}{c_v} = \frac{7}{5} \quad , \quad \text{R}\sim 287 \ J/K \cdot Kg$$

$$\because c \neq c(k) \ \Rightarrow Non-dispersive\ \ waves$$

(2) Pure 2-D internal gravity wave



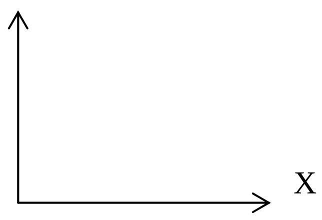
$$\nu = \frac{-Nk}{(k^2 + m^2)} \ \Rightarrow c = \frac{\nu}{k} = -\frac{N}{k^2 + m^2}$$

$$\because c_{gx} = \frac{\partial \nu}{\partial k} \quad c_{gy} = \frac{\partial \nu}{\partial m}$$

$$\Rightarrow dispersive\ wave$$

圖 10-5

(3) 2-D Rossby wave



$$\nu = \frac{-\beta k}{(k^2 + l^2)} \quad 其中 \ \beta(\text{科氏參數}) = \frac{df}{dy}$$

$$又 \quad f = 2\Omega \sin\phi$$

$$\because c_{gx} = \frac{\partial \nu}{\partial k} \ , \ c_{gy} = \frac{\partial \nu}{\partial l}$$

$$\Rightarrow dispersive\ \ wave$$

圖 10-6