Chapter 8 Partial Differential Equations (P.D.E)

A partial differential equation is a differential equation involving several (more than one) independent variables and their partial derivatives.

For Example:

(1) $\frac{\partial^3 u}{\partial x^3} + \left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial^2 u}{\partial x^2}$, a P.D.E for the function $u(x,t)$ (3rd-order in this case)

(2) $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$ ← a system of P.D.Es for two functions $u(x,y)$ and $v(x,y)$. Note that it is called the Cauchy Rieman equation which is used for 2D potential flow.

(3) $\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x \partial t} + u$, a 2nd-order PDE for $u(x,t)$.

Three common PDEs:

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow$ heat equation where $u(x,t)$ = temperature

$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow$ wave equation where $u(x,t)$ = vertical displacement of a string at point x at the time t.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$ Laplace's equation where $u(x,y)$ is a potential function.

§ Using "separation of variables" to solve the heat equation.

$$\left.\begin{aligned} &\text{PDE}: \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \\ &\text{IC}: u(x,0) = f(x), \quad 0 < x < l \\ &\text{BC}: u(0,t) = f(l,t) = 0 \end{aligned}\right\} \quad (8.1) \quad \text{find } u(x,t) = ?$$

Using the separation of variable technique, i.e., assume

$$u(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{\partial u}{\partial t} = XT' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T \qquad (8.2)$$

Substituting (8.2) into the PDE in (8.1):

$$XT' = \alpha^2 X''T \qquad (8.3)$$

$$\frac{(8.3)}{\alpha^2 XT} \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha^2 T} \qquad (8.4)$$

The LHS of (8.4) is a function of *x*, and the RHS of (8.4) is a function of *t*. If Eq.(8.4) is a valid for all *x* and *t*, this requires:

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = \text{constant} = -\lambda \qquad (8.5)$$

Check with BCs:

$$\left.\begin{aligned} 0 &= u(0,t) = X(0)T(t) \quad (8.6) \\ 0 &= u(l,t) = X(l)T(t) \quad (8.7) \end{aligned}\right\}$$

$$\Rightarrow X(0) = X(l) = 0 \qquad (8.8)$$

Thus Eq. (8.5) implies two equations for $X(x)$ and $T(t)$

$$X'' + \lambda X = 0,\ X(0) = 0,\ X(l) = 0 \qquad (8.9)$$

and

$$T' + \lambda\alpha^2 T = 0 \qquad (8.10)$$

Eq.(8.9) $\Rightarrow$ It has a nontrivial solution $X(x)$ only if $\lambda = \lambda_n = \frac{n^2\pi^2}{l^2},\quad n = 1, 2, \cdots$

and $X(x) = X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$

Eq.(8.10) $\Rightarrow$ $T(t) = T_n(t) = e^{-\alpha^2\lambda_n t} = e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$

Thus $u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$

$\therefore\ u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t}$ (c_n=coeff. to be determined)

Checking with IC: $u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$ (8.11)

$$\int_0^l (8.11)\sin\left(\frac{m\pi x}{l}\right)dx \Rightarrow \int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \sum_{n=1}^{\infty} c_n \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx$$

$$\because \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} 0, & n \neq m \\ \frac{l}{2}, & n = m \end{cases}$$

Recall Ch. 7 Fourier Series

$$\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx = c_m \frac{l}{2}$$

$$\Rightarrow c_m = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx \quad \text{or} \quad c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad (8.12)$$

Hence, the solution $u(x,t)$ to the PDE in (8.1) is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}t}$$

$$\text{where } c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad (8.13)$$

Ex: A thin aluminum bar ($\alpha^2 = 0.86\ \text{cm}^2\text{s}^{-1}$) which is 10 cm long is heated to a uniform temperature of 100°C. At time $t = 0$, the ends of the bar are plunged into an ice both at 0°C, and thereafter they are maintained at this temperature. No heat is allowed to escape through the lateral surface of the bar. Find an expression for the temperature at any point in the bar at any later time t.

Sol:

Let $u(x,t)$ be the temperature (in°C) in the bar at the point x at the time t. Then $u(x,t)$ satisfies this PDE:

$$\frac{\partial u}{\partial t} = 0.86\frac{\partial^2 u}{\partial x^2}, \quad \begin{cases} u(x,0) = 100 & , 0 < x < 10 \\ u(0,t) = u(10,t) = 0 \end{cases}$$

Using Eq. 13, the solution to this problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{0.86 n^2 \pi^2}{100}t}$$

where $c_n = \frac{2}{10}\int_0^{10} 100\sin\left(\frac{n\pi x}{10}\right)dx$

$$= 20\cdot\frac{(-10)}{n\pi}\left[\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= -\frac{200}{n\pi}\left[\cos(n\pi)-1\right] = \frac{200}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \begin{cases} 0, & n \text{ even} \\ \frac{400}{n\pi}, & n \text{ odd} \end{cases}$$

$$\Rightarrow u(x,t) = \frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)}\sin\left[\frac{(2n+1)\pi x}{10}\right]\exp\left[-\frac{0.86(2n+1)^2\pi^2}{100}t\right]$$

Ex: Consider a thin metal rod of length l and thermal diffusivity α^2, whose sides and ends are insulated so that there is no passage of heat through them. Let the initial temperature distribution in the rod be $f(x)$. Find the temperature distribution in the rod at any later time t.

Sol:

Let $u(x,t)$ be the temperature in the rod at any point x at the time t.

$$\left.\begin{array}{l} \text{PDE}: \ \frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \\ \text{IC}: \ u(x,0) = f(x),\ 0 < x < l \\ \text{BCs}: \ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0 \end{array}\right\} \qquad (8.14)$$

Assume

$$u(x,t) = X(x)T(t)$$

$$\Rightarrow \frac{\partial u}{\partial t} = XT' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting into the PDE

$$\Rightarrow \frac{1}{\alpha^2 XT}\left\{XT' = \alpha^2 X''T\right\} \Rightarrow \frac{X''}{X} = \frac{T'}{\alpha^2 T} = -\lambda \quad \text{(say !)}$$

Thus we get two separate equations for X and T

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda\alpha^2 T = 0$$

Substituting into BCs

$$0 = \frac{\partial u}{\partial x}(0,t) = X'(0)T(t), \quad 0 = \frac{\partial u}{\partial x}(l,t) = X'(l)T(t)$$

$$\Rightarrow X'(0) = 0 = X'(l)$$

Thus the governing equation for $X(x)$ is

$$X'' + \lambda X = 0, X'(0) = 0, \ X'(l) = 0$$

$\Rightarrow$ It has a nontrivial solution $X(x)$ only if $\lambda = \lambda_n = \dfrac{n^2\pi^2}{l^2}, \quad n = 0, 1, 2, \cdots$

$$X(x) = X_n(x) = \cos\left(\frac{n\pi x}{l}\right)$$

And the solution for $T(t)$ is

$$T(t) = e^{-\lambda\alpha^2 t} = e^{-\frac{\alpha^2 n^2 \pi^2}{l^2} t}$$

Hence

$$u_n(x,t) = X(x)T(t) = \cos\left(\frac{n\pi x}{l}\right)\exp\left[-\frac{\alpha^2 n^2 \pi^2}{l^2} t\right]$$

The solution is the linear combination of $u_n(x,t)$

$$u(x,t) = f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right)\exp\left[-\frac{\alpha^2 n^2 \pi^2}{l^2} t\right]$$

using IC to decide $c_n \Rightarrow$

$$u(x,0) = f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right) \qquad 0 \le x \le l$$

Thus we must expand $f(x)$ is a Fourier cosine series on the interval $0 \le x \le 1$.
Then, according to the theorem in Fourier series,

$$c_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 0, 1, 2, ...$$

Hence, the solution $u(x,t)$ is

$$u(x,t) = \frac{1}{l}\int_0^l f(x)dx + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right)\exp\left[-\frac{\alpha^2 n^2 \pi^2}{l^2} t\right] \qquad (8.15)$$

and c_n is given by above formula.

§ The Wave Equation.

$$\left.\begin{array}{ll} \text{PDE}: & \dfrac{\partial^2 u}{\partial t^2} = C^2 \dfrac{\partial^2 u}{\partial x^2} \\ \text{ICs}: & \begin{array}{l} u(x,0) = f(x) \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{array}, \quad 0 \le x \le l \\ \text{BCs}: & u(0,t) = u(l,t) = 0 \end{array}\right\} \qquad (8.16)$$

Again, we can solution the wave equation in (8.16) by the method of "separation of variables". So Let $u(x,t) = X(x)T(t)$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = XT'' \text{ and } \frac{\partial^2 u}{\partial x^2} = X''T$$

Substituting in to the PDE $\Rightarrow \dfrac{1}{c^2 XT}\left\{XT'' = c^2 X''T\right\}$

$$\Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda = \text{constant} \qquad (8.17)$$

Also the BCs become

$$\left.\begin{array}{l} 0 = u(0,t) = X(0)T(t) \\ 0 = u(l,t) = X(l)T(t) \end{array}\right\} \quad X(0) = 0, \quad X(l) = 0$$

So the governing equation for $X(x)$ is

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(l) = 0$$

$$\Rightarrow X(x) = X_n(x) = \sin\left(\frac{n\pi x}{l}\right), \text{ i.e., } \lambda = \lambda_n = \frac{n^2\pi^2}{l^2}$$

and the governing equation for $T(t)$ is

$$T'' + \lambda c^2 T = 0 \Rightarrow T(t) = T_n(t) = a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)$$

Hence

$$u_n(x,t) = X_n(x)T_n(t)$$

$$= \sin\left(\frac{n\pi\pi}{l}\right)\left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right]$$

Thus

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\pi}{l}\right)\left[a_n \cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right] \qquad (8.18)$$

Using ICs to find a_n and b_n $\Rightarrow$

$$u(x,0)=f(x)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$
$$\frac{\partial u}{\partial t}(x,0)=g(x)=\sum_{n=1}^{\infty}\left(\frac{n\pi c}{l}\right)b_n \sin\left(\frac{n\pi x}{l}\right)$$

Thus a_n and b_n can be determined using the Fourier series theorem

$$a_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$
$$b_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad (8.19)$$

Note that for the special case when $g(x)=0$, that is, the string is released with zero initial velocity. Then the displacement $u(x,t)$ of the string at any time at $t>0$ is given by:

$$u(x,t)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right), \quad a_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad (8.20)$$

The first term (n=1) in Eq.(8.20) represents the first mode of vibration in which the string oscillates about it's equilibrium position with frequency:

$$\omega_1=\frac{1}{2\pi}\frac{1}{T_1}=\frac{1}{2\pi}\frac{\pi c}{l}=\frac{c}{2l} \qquad \text{(cycles per second)}$$

(first harmonic of the string or the fundamental frequency)

Similarly, the frequency for the nth mode is

$$\omega_n=\frac{1}{2\pi T_n}=\frac{1}{2\pi}\frac{n\pi c}{l}=\frac{nc}{2l}=n\omega_1 \qquad \text{(cycles per second)}$$

$\uparrow$
(nth harmonic of the string)

Note that using the trigometric identity

$$\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)=\frac{1}{2}\left\{\sin\left[\frac{n\pi}{l}(x-ct)\right]+\sin\left[\frac{n\pi}{l}(x+ct)\right]\right\} \qquad (8.21)$$

Let $F(x)$ be odd periodic extension of $f(x)$ on the interval $-l<x<l$; that is

$$F(x)=\begin{cases} f(x) & 0<x<l \\ -f(-x) & -l<x<0 \end{cases} \quad \text{and} \quad F(x+2l)=F(x)$$

Then the Fourier series for $F(x)$ is

$$F(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) \qquad \left(= f(x),\ 0 < x < l\right)$$

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad \left(= a_n,\ 0 < x < l\right)$$

Therefore, we can rewrite $u(x,t)$ by combing (8.20) and (8.21) as

$$u(x,t) = \frac{1}{2}\left[\underbrace{F(x-ct)}_{\text{a wave moving with velocity } c \text{ in the positive } x \text{ direction}} + \underbrace{F(x+ct)}_{\text{a wave moving with velocity } c \text{ in the negative } x \text{ direction}}\right] \qquad (8.22)$$

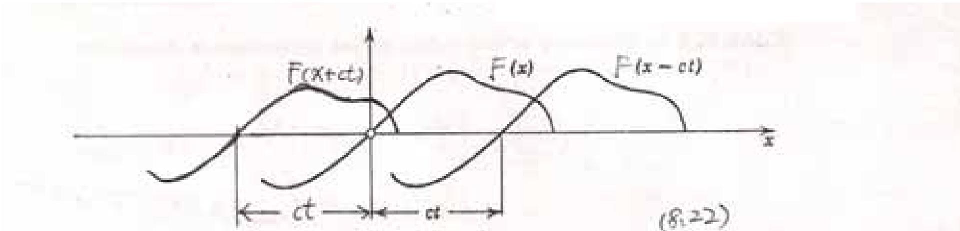
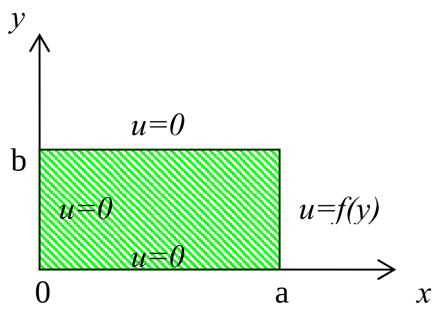


Fig. 247. Interpretation of (8.22)

§ The Laplace Equation

Ex: Find a function $u(x,y)$ which satisfies the Laplace's equation in the rectangle $0<x<a$, $0<y<b$, and which also satisfies the boundary conditions.



PDE: $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ (8.23)

BCs: $\begin{cases} u(x,0)=0, & u(x,b)=0 \\ u(0,y)=0, & u(a,y)=f(y) \end{cases}$

(Dirichlet-type BC)

Sol: Using the method of "separation of variables",

Let $u(x,y)=X(x)Y(y)$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=X''Y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2}=XY''$$

Substituting into the Laplace's equation in (8.23)

$$\frac{1}{XY}\{X''Y+XY''=0\}$$
$$\Rightarrow \frac{Y''}{Y}=-\frac{X''}{X}=-\lambda=\text{constant} \tag{8.24}$$

The BCs becomes

$$0=u(x,0)=X(x)Y(y),\quad 0=u(x,b)=X(x)Y(b)$$
$$0=u(0,y)=X(0)Y(y),$$
$$\Rightarrow Y(0)=Y(b)=X(0)=0$$

Hence $u(x,y)=X(x)Y(y)$ is a solution to (8.23) if

$$Y''+\lambda Y=0,\quad Y(0)=0,\quad Y(b)=0 \tag{8.25}$$

and

$$X''-\lambda X=0,\quad X(0)=0 \tag{8.26}$$

Eq. (8.25) has a nontrivial solution $Y(y)$ only if $\lambda=\lambda_n=\dfrac{n^2\pi^2}{b^2}$ and

$$Y(y)=Y_n(y)=\sin\left(\frac{n\pi\, y}{b}\right) \tag{8.27}$$

Eq. (8.26) implies that

$$X_n(x) = \sinh\left(\frac{n\pi x}{b}\right)$$

Thus

$$u_n(x,y) = X_n(x)Y_n(y) = \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

The general solution is

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad (8.28)$$

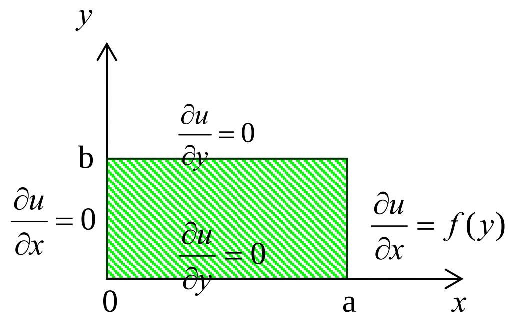
Using the last BC to decide c_n :

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right), \qquad 0 < y < b$$

So we must expand $f(y)$ in a Fourier sine series on the interval 0<*y*<*b*. Thus according to the theorem in Fourier series:

$$c_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy, \qquad n = 1, 2, 3, \cdots \qquad (8.29)$$

Ex: Find a function $u(x,y)$ which satisfies the Laplace's equation in the rectangle 0<*x*<*a*, 0<*y*<*b*, and also satisfies the boundary conditions.



PDE: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \qquad (8.30)$

BCs: $\begin{cases} \dfrac{\partial u}{\partial y}(x,0) = 0, \ \dfrac{\partial u}{\partial y}(x,b) = 0 \\ \dfrac{\partial u}{\partial x}(x,0) = 0, \ \dfrac{\partial u}{\partial x}(x,b) = f(y) \end{cases}$

(Reumann-type BC)

Sol: Again Let $u(x,y) = X(x)Y(y)$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = X''Y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

Substituting into the Laplace Eq.(8.30)

$$\frac{1}{XY}\{X''Y + XY'' = 0\}$$

$$\Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} = -\lambda = \text{constant}$$

The BCs become

$$0 = \frac{\partial u}{\partial y}(x,0) = X(x)Y'(0), \quad 0 = \frac{\partial u}{\partial y}(x,b) = X(x)Y'(b)$$

$$0 = \frac{\partial u}{\partial x}(0,y) = X'(0)Y(y),$$

$$\Rightarrow Y'(0) = 0, \quad Y'(b) = 0, \quad X'(0) = 0$$

Hence $u(x,y) = X(x)Y(y)$ is a solution to (8.30) if

$$Y'' + \lambda Y = 0, \quad Y'(0) = 0, \quad Y'(b) = 0 \qquad (8.31)$$

and

$$X'' - \lambda X = 0, \quad X'(0) = 0 \qquad (8.32)$$

Eq. (8.31) has a nontrivial solution *Y(y)* only if $\lambda = \lambda_n = \frac{n^2\pi^2}{b^2}$ and n $= 1,2,3......$

$$Y(y) = Y_n(y) = \cos\left(\frac{n\pi y}{b}\right)$$

Eq. (8.32) implies that

$$X(x) = X_n(x) = \cosh\left(\frac{n\pi x}{b}\right)$$

Thus

$$u_n(x,y) = X_n(x)Y_n(y) = \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Then the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi x}{b}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad (8.33)$$

And c_n can be determined by the last BC:

$$\because \frac{\partial u}{\partial x}(a,y) = f(y)$$

$$\left.\frac{\partial(8.33)}{\partial x}\right|_{x=a} \Rightarrow f(y) = \sum_{n=1}^{\infty}\left(\frac{n\pi}{b}\right)c_n \sinh\left(\frac{n\pi a}{b}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad (8.34)$$

According to the theorem in Fourier series, we can expand $f(y)$ in the cosine series as

$$f(y) = \frac{1}{b}\int_0^b f(y)dy + \frac{2}{b}\sum_{n=1}^{\infty}\left[\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]\cos\left(\frac{n\pi y}{b}\right) \qquad (8.35)$$

Comparing Eq. (8.34 with Eq. (8.35, we know this condition

$$\int_0^b f(y)dy = 0 \qquad (8.36)$$

is necessary for this Neumann problem to have a solution.

Then coefficients c_n is

$$c_n = \frac{2}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\cos\left(\frac{n\pi y}{b}\right)dy, \qquad \mathrm{n} \geq 1 \qquad (8.37)$$

Note that c_0 in (8.33) remains arbitrary. Thus the solution $u(x, y)$ is

determined only up to an arbitrary constant. This is a property of all Neumann problems.