Chapter 7 Fourier Series

Let $f(x)$ be defined on the interval $-l \le x \le l$, then $f(x)$ could be expanded by an infinite Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right] \qquad (7.1)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 0, 1, 2, \ldots \qquad (7.2)$$

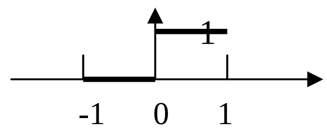
$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad n = 0, 1, 2, \ldots \qquad (7.3)$$

Theorem: Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $-l \le x \le l$.

Compute the numbers a_n and b_n from (7.2) and (7.3) form the infinite series (7.1). This series, called the Fourier Series for the interval $-l \le x \le l$, converges to $f(x)$ if $f(x)$ is continuous at x, and to $1/2\left[f(x^+) + f(x^-)\right]$ if $f(x)$ is discontinuous at x. At $x = \pm l$, the Fourier series (7.1) converge to $1/2\left[f(l) + f(-l)\right]$, where $f(\pm l)$ is the limit of $f(x)$ as x approaches $\pm l$.

Ex: Let $f(x)$ be zero for $-1 \le x < 0$ and 1 for $0 < x \le 1$. Compute the Fourier series for $f(x)$ on the interval $-1 \le x \le 1$.

Sol:



$$a_0 = \int_{-1}^{1} f(x)dx = \int_0^1 (1)dx = 1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_0^1 \cos(n\pi x)dx = \frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 = 0, \quad n \ge 1$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f((x))\sin(n\pi x)dx = \int_0^1 \sin(n\pi x)dx = -\frac{1}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{1}{n\pi}\left[1 - \cos(n\pi)\right] = \frac{1-(-1)^n}{n\pi} = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{n\pi} & n \text{ odd} \end{cases}$$

Thus, the Fourier series for $f(x)$ on the interval $-1 \le x \le 1$ is

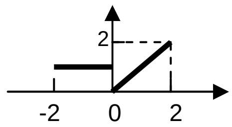
$$f((x)) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi x) + \frac{2}{3\pi}\sin(3\pi x) + \frac{2}{5\pi}\sin((5\pi x) + \cdots$$

Ex: Let $f(x)$ be 1 for $-2 \le x < 0$ and x for $0 \le x \le 2$. Compute the Fourier series for $f(x)$ on the interval $-2 \le x \le 2$.

Sol: Since l=2 in this problem.

$$\Rightarrow a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0}(1)dx + \frac{1}{2}\int_{0}^{2} xdx = \frac{1}{2}\left[2 + \frac{x^2}{2}\bigg|_0^2\right] = \frac{1}{2}[2+2] = 2$$

$$\Rightarrow a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$



$$= \underbrace{\frac{1}{2}\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx}_{} + \underbrace{\frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx}_{}$$

$$= \frac{1}{2}\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\bigg|_{-2}^{0}$$

$$= \frac{1}{n\pi}(0-0) = 0$$

$$\Rightarrow x = u, \quad dv = \cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow dx = du, \quad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{1}{2}\left[x\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \frac{1}{2}\frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= 0 - \frac{1}{n\pi}\left(\frac{-2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\bigg|_0^2$$

$$= \frac{2}{n^2\pi^2}[\cos(n\pi) - 1]$$

$$\Rightarrow a_n = \frac{2}{n^2\pi^2}[\cos(n\pi) - 1], \qquad \text{n} \ge 1$$

And

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \underbrace{\frac{1}{2}\int_{-2}^{0}(1)\sin\left(\frac{n\pi x}{2}\right)dx}_{} + \underbrace{\frac{1}{2}\int_{-0}^{2} x\sin\left(\frac{n\pi\pi}{2}\right)dx}_{}$$

$$u = x, \quad dv = \sin\left(\frac{n\pi\pi}{2}\right)dx$$

$$= \frac{1}{2}\left(-\frac{2}{n\pi}\right)\left[\cos\left(\frac{n\pi x}{2}\right)\right]\bigg|_{-2}^{0}$$

$$= \frac{-1}{n\pi}[1 - \cos(-n\pi)]$$

$$= \frac{-1}{n\pi}[1 - \cos(n\pi)]$$

$$= \frac{1}{2}\left[-x\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \frac{1}{2}\frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi}\cos(n\pi)\right] + \frac{1}{n\pi}\frac{2}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}\cos(n\pi) + 0$$

$$\Rightarrow b_n = -\frac{1}{n\pi}[1-\cos(n\pi)] - \frac{2}{n\pi}\cos(n\pi) = -\frac{1}{n\pi}[1+\cos(n\pi)]$$

Thus,

$$a_n = \frac{2}{n^2\pi^2}[\cos(n\pi)-1] = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n^2\pi^2}, & n \text{ odd} \end{cases}$$

$$\mathrm{b_n} = \frac{-1}{n\pi}[1+\cos(n\pi)] = \begin{cases} \frac{-2}{n\pi}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Then the Fourier series for *f(x)* on the interval $-2 \le x \le 2$ is

$$f(x) = 1 - \frac{4}{\pi^2}\cos\left(\frac{\pi x}{2}\right) - \frac{1}{\pi}\sin(\pi x) - \frac{4}{9\pi^2}\cos\left(\frac{3\pi x}{2}\right) - \frac{1}{2\pi}\sin(2\pi x) + \cdots$$
$$= 1 - \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos\left[\frac{(2m+1)\pi x}{2}\right] - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m}\sin(m\pi x) \qquad (7.4)$$

Note that according to the theorem, this Fourier series converges to 1 if $-2 < x < 0$; it converges to x if $0 < x < 2$; it converges to $1/2(1+0) = 1/2$ at $x = 0$. Then at $x = 0$, the Fourier series in (7.4) is

$$1 - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] \xrightarrow{converge} \frac{1}{2}$$
$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Theorem: a function $f(x)$ can be expanded in one, and only one, the Fourier series on the interval $-l \le x \le l$.

Proof: Suppose that $f(x)$ is piecewise continuous and can be written as

$$f(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty}\left[c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right)\right] \qquad (7.5)$$

for some numbers c_k and d_k.

Assume Eq. (7.5) is valid for all points in the interval $-l \le x \le l$.

$$\int_{-l}^{l}(7.5)dx \Rightarrow \int_{-l}^{l} f(x)dx = \int_{-l}^{l}\frac{c_0}{2} + \sum_{k=1}^{\infty}\left[c_k\int_{-l}^{l}\cos\left(\frac{k\pi x}{l}\right)dx + d_k\int_{-l}^{l}\sin\left(\frac{k\pi x}{l}\right)dx\right]$$

$$= c_0 l \qquad = \frac{l}{k\pi}\left[\sin\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} = 0 \qquad = \frac{-l}{k\pi}\left[\cos\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} = 0$$

$$\Rightarrow c_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$\int_{-l}^{l}(7.5)\cos\left(\frac{n\pi x}{l}\right)dx \Rightarrow$$

$$\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \underbrace{\frac{c_0}{2}\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx}_{=0} + \sum_{k=1}^{\infty}\Bigg[c_k \underbrace{\int_{l}^{-l}\cos\left(\frac{k\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx}_{=\begin{cases}0 & \text{k} \neq \text{n}\\ l & \text{k} = n\end{cases}}$$

$$+ \underbrace{d_k\int_{l}^{-l}\sin\left(\frac{k\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx}_{=0}\Bigg]$$

$$\because \int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)dx = \frac{l}{n\pi}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} = 0 - 0 = 0$$

$$\int_{-l}^{l}\cos\left(\frac{k\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = \begin{cases}0 & , k \neq n\\ l & , k = n\end{cases} \qquad (7.6)$$

$$\int_{l}^{-l}\sin\left(\frac{k\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = 0 \qquad (7.7)$$

$$\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = l\cdot c_n$$

$$\Rightarrow c_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Similarly, $\int_{l}^{-l}(5)\sin\left(\frac{n\pi x}{l}\right)dx \Rightarrow$

$$\int_{l}^{-l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \underbrace{\frac{c_0}{2}\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)dx}_{=0} + \sum_{k=1}^{\infty}\Bigg[\underbrace{c_k\int_{-l}^{l}\cos\left(\frac{k\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx}_{=0}$$

$$+ \underbrace{d_k\int_{-l}^{l}\sin\left(\frac{k\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx}_{=\begin{cases}0, & k \neq n\\ l, & k = n\end{cases}}\Bigg]$$

$$\because \int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)dx = -\frac{l}{n\pi}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_{-l}^{l} = 0$$

$$\int_{-l}^{l}\cos\left(\frac{k\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = 0$$

$$\int_{l}^{-l}\sin\left(\frac{k\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \begin{cases}0, & k \neq n\\ l, & k = n\end{cases} \qquad (7.8)$$

$$\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = d_n \cdot l$$

$$\Rightarrow d_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Thus, coefficients c_n and d_n must equal the Fourier series coefficients a_n and b_n.

Note that using the triangle identity,

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{k\pi x}{l}\right)dx = \frac{1}{2}\left[\int_{-l}^{l}\cos\left(\frac{(n+k)\pi x}{l}\right)dx + \int_{-l}^{l}\cos\left(\frac{(n-k)\pi x}{l}\right)dx\right]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\frac{l}{(n+k)\pi}\left\{\sin\left[\frac{(n+k)\pi x}{l}\right]\right\}_{-l}^{l} + \begin{cases} \frac{l}{2(n-k)\pi}\left\{\sin\left[\frac{(n-k)\pi x}{l}\right]\right\}_{-l}^{l} & \text{if } n \neq k \\ \frac{1}{2}\int_{-l}^{l} dx & \text{if } n = k \end{cases}$$

$$= \begin{cases} 0, \text{ n} \neq \text{k} \\ \text{l}, \text{ n} = \text{k} \end{cases} \qquad (7.6)$$

And $$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{k\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left[\sin\left(\frac{(k+n)\pi x}{l}\right)+\sin\left(\frac{(k-n)\pi x}{l}\right)\right]dx$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

$$= -\frac{1}{2}\frac{l}{(n+k)\pi}\left\{\cos\left[\frac{(n+k)\pi x}{l}\right]\right\}_{-l}^{l} + \begin{cases} \frac{-l}{2(k-n)\pi}\left\{\cos\left[\frac{(k-n)\pi x}{l}\right]\right\}_{-l}^{l} & \text{if } k \neq n \\ \frac{1}{2}\int_{-l}^{l} 0\,dx = 0 & \text{if } k = n \end{cases}$$

$$= 0 \qquad (7.7)$$

Also $$\int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{k\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{\cos\left[\frac{(n-k)\pi x}{l}\right]-\cos\left[\frac{(n+k)\pi x}{l}\right]\right\}dx$$

$$\sin A\sin B = \frac{1}{2}[\cos(A+B)-\cos(A-B)]$$

$$= \begin{cases} \frac{l}{2(n-k)\pi}\sin\left[\frac{(n-k)\pi x}{l}\right]_{-l}^{l} & \text{if } n \neq k \\ \frac{1}{2}\int_{-l}^{l}(1)dx & \text{if } n = k \end{cases} - \frac{l}{2(n+k)\pi}\sin\left[\frac{(n+k)\pi x}{l}\right]_{-l}^{l}$$

$$= \begin{cases} 0, & n \neq k \\ l, & n = k \end{cases} \qquad (7.8)$$

Note that the functions $\cos\left(\frac{n\pi x}{l}\right)$ and $\sin\left(\frac{n\pi x}{l}\right)$, n=1,2,3,…., are periodic with the period of $2l$.

$$\Rightarrow \cos\left[\frac{n\pi}{l}(x+2l)\right]=\cos\left[\frac{n\pi x}{l}+2n\pi\right]=\cos\left(\frac{n\pi x}{l}\right)$$

Similarly

$$\Rightarrow \sin\left[\frac{n\pi}{l}(x+2l)\right]=\sin\left[\frac{n\pi x}{l}+2n\pi\right]=\sin\left(\frac{n\pi x}{l}\right)$$

§ **Fourier Series: Even and Odd Functions.**

Def: A function $f(x)$ is an <u>even</u> function if $f(-x)=f(x)$
A function $f(x)$ is an <u>odd</u> function if $f(-x)=-f(x)$

Ex: $f(x)=x^2$ is even $\because f(-x)=(-x)^2=x^2=f(x)$

$f(x)=\cos(x)$ is even $\because f(-x)=\cos(-x)=\cos x=f(x)$
$f(x)=\sin(x)$ is odd $\because f(-x)=\sin(-x)=-\sin x=-f(x)$

Property:

1. "even function" $\times$ "even function"= "even function"
2. "odd function" $\times$ "odd function"= "even function"
3. "even function" $\times$ "odd function"= "odd function"
4. $\int_{-l}^{l} f(x)dx=0$ if $f(x)$ is odd (7.9)
5. $\int_{-l}^{l} f(x)dx=2\int_{0}^{2} f(x)dx$ if $f(x)$ is even (7.10)

<u>Lemma</u>:
(a) The Fourier series for an <u>even</u> function is a pure <u>cosine</u> series; it contains no terms of the form $\sin\left(\frac{n\pi x}{l}\right)$.
(b) The Fourier series for an <u>odd</u> function is a pure <u>sine</u> series; it contains no terms of the form $\cos\left(\frac{n\pi x}{l}\right)$.

Theorem: Let $f(x)$ and $f'(x)$ be piecewise continuous on the interval $0 \le x \le l$. Then, on this interval, $f(x)$ can be expanded in either a pure cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \; n = 0, 1, 2, \cdots \quad (7.11)$$

or a pure sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \; b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \; n = 1, 2, \cdots \quad (7.12)$$

Ex: Expand the function $f(x) = 1$ in a pure sine series on the interval $0 < x < \pi$.

Sol: Using the above theorem,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \text{ and } l = \pi$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} (1)\sin(nx)dx = \frac{-2}{n\pi}\left[\cos(nx)\right]_0^{\pi}$

$$= \frac{2}{n\pi}\left[1 - \cos(n\pi)\right] = \begin{cases} 0 & , n \text{ even} \\ \frac{4}{n\pi} & , n \text{ odd} \end{cases}$$

$$\Rightarrow f(x) = 1 = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi}\sin((2m+1)x)$$

$$= \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right], \qquad 0 < x < \pi$$

$$\Rightarrow \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots = \frac{\pi}{4}, \quad 0 < x < \pi$$

Ex: Expand the function $f(x) = e^x$ in a pure cosine series on the interval $0 \le x \le 1$

Sol: Using the above theorem,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

where $a_0 = 2\int_0^1 e^x dx = 2\left[e^x\right]_0^1 = 2(e-1)$

$$a_n = 2\int_0^1 e^x \cos(n\pi x)dx = 2 \cdot R_e \int_0^1 e^x e^{in\pi x} dx$$

$$= 2R_e \int_0^1 e^{(1+in\pi)x} dx = 2R_e \left[\frac{e^{(1+in\pi)x}}{1+in\pi} \right]_0^1$$

$$= 2R_e \left[\frac{e^{1+in\pi} - 1}{1+in\pi} \right] = 2R_e \left[\frac{\left(e^{1+in\pi} - 1\right)(1-in\pi)}{(1+in\pi)(1-in\pi)} \right].$$

$$= 2R_e \left[\frac{e^{1+in\pi} - 1 - in\pi e^{1+in\pi} + in\pi}{1+n^2\pi^2} \right]$$

$$= 2R_e \left[\frac{e(\cos n\pi + i\sin n\pi) - 1 - in\pi e(\cos n\pi + i\sin n\pi) + in\pi}{1+n^2\pi^2} \right]$$

$$= \frac{2(e\cos n\pi - 1)}{1+n^2\pi^2}$$

$$\Rightarrow f(x) = e^x = e - 1 + 2\sum_{n=1}^{\infty} \frac{(e\cos n\pi - 1)}{1+n^2\pi^2} \cos(n\pi x), \quad 0 \le x \le 1$$

§ The Fourier Integral

Recall that the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right], \quad -l < x < l \qquad (7.1)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \quad n = 0, 1, 2, 3, \cdots \qquad (7.2)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad n = 1, 2, 3, \cdots \qquad (7.3)$$

The figure below (Fig.239) shows the periodic function of a period of $2l$,

$$f_{2l}(x) = e^{-|x|} \text{ when } -l < x < l, \quad \text{and } f_{2l}(x+2l) = f_{2l}(x)$$

When $l = \infty$, we then obtain a function $f(x)$ which is no longer periodic

$$f(x) = \lim_{l\to\infty} f_{2l}(x) = e^{-|x|}$$

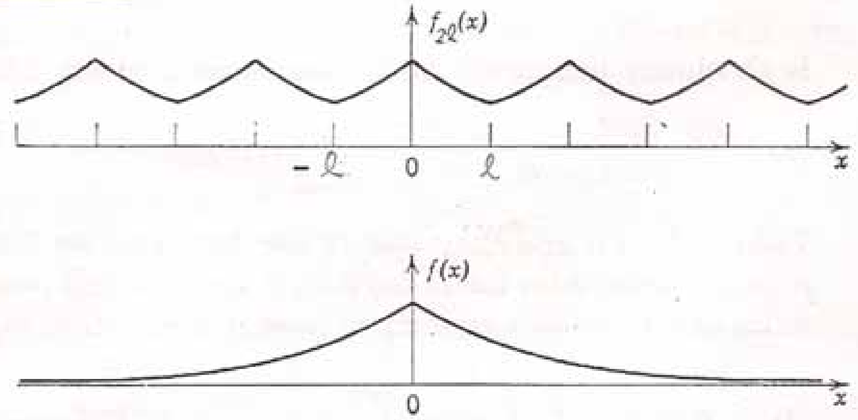


Fig. 239. Example 2

We now consider any periodic function $f_{2l}(x)$ of period (or wavelength in this case) $2l$ which can be represented by a Fourier series. Let $k_n = \frac{2n\pi}{2l} = \frac{n\pi}{l}$, then the Fourier-series expansion of $f_{2l}(x)$ can be written as,

$$f_{2l}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(k_n x) + b_n \sin(k_n x) \right].$$

We can insert a_n and b_n into the above formula, using the variable of integration by v. Then the Fourier series of $f_{2l}(x)$ becomes

$$f_{2l}(x) = \frac{1}{2l} \int_{-l}^{l} f_{2l}(v) dv + \frac{1}{l} \sum_{n=1}^{\infty} \left[\cos(k_n x) \int_{-l}^{l} f_{2l}(v) \cos(k_n v) dv + \sin(k_n x) \int_{-l}^{l} f_{2l}(v) \sin(k_n v) dv \right.$$

$$\Delta k = k_{n+1} - k_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l} \Rightarrow \frac{1}{l} = \frac{\Delta k}{\pi}$$

Thus the above Fourier series can be rewritten as

$$f_{2l}(x) = \frac{1}{2l} \int_{-l}^{l} f_{2l}(v) dv + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\cos(k_n x) \Delta k \int_{-l}^{l} f_{2l}(v) \cos(k_n v) dv \right.$$
$$\left. + \sin(k_n x) \Delta k \int_{-l}^{l} f_{2l}(v) \sin(k_n v) dv \right] \qquad (7.13)$$

This representation in (7.13) is valid for any fixed l, arbitrarily large, but still finite.

We now let $l \to \infty$ and assume the resulting nonperiodic function $f(x) = \lim_{2l \to \infty} f_{2l}(x)$ is integrable on the x-axis; that is, the following integral exists:

$$\int_{-\infty}^{\infty} |f(x)| dx \qquad (7.14)$$

Then $\frac{1}{l} \to 0$ and the first term of the RHS of (7.13) vanishes. Also $\Delta k = \frac{\pi}{l} \to 0$ and the infinite series in (7.13) becomes an integral from 0 to ∞. Thus $f(x)$ in (7.13) can be written as,

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos(kx)\underbrace{\int_{-\infty}^{\infty} f(v)\cos(kv)dv}_{\equiv A(k) \quad (7.16)} + \sin(kx)\underbrace{\int_{-\infty}^{\infty} f(v)\sin(kv)dv}_{\equiv B(k) \quad (7.17)}\Big]dk \qquad (7.15)$$

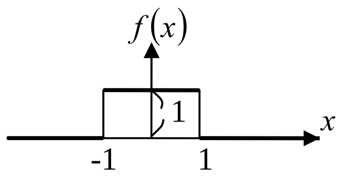
$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^{\infty}\left[A(k)\cos(kx) + B(k)\sin(kx)\right]dk \qquad (7.18)$$

Eq. (7.18) is a representation of $f(x)$ by a Fourier integral.

Theorem: if $f(x)$ is piecewise continuous in every finite interval and has a right-hand derivative and a left-hand derivative at every point and $\int_{-\infty}^{\infty} |f(x)| dx$ exists, then $f(x)$ can be represented by a Fourier integral. At a point where $f(x)$ is discontinuous, the value of the Fourier integral equals the average of the left- and light- hand limits of $f(x)$ at that point.

Ex: Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1 \end{cases}$$



Sol: Form (7.16) and (7.17)

$$A(k) = \int_{-\infty}^{\infty} f(v)\cos(kv)dv = \int_{-1}^{1}\cos(kv)dv = \left.\frac{\sin(kv)}{k}\right|_{v=-1}^{v=1} = \frac{2\sin k}{k}$$

$$B(k) = \int_{-1}^{1} f(v)\sin(kv)dv = \int_{-1}^{1}\sin(kv)dv = \left.\frac{-\cos(kv)}{k}\right|_{v=-1}^{v=1} = 0$$

Thus by (7.18),

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(k)\cos(kx)dk = \frac{2}{\pi}\int_0^{\infty}\frac{\sin k\cos(kx)}{k}dk \qquad (7.19)$$

$$\left.\begin{array}{r} \text{At } x = 1, \text{left-hand limit of } f(x) = 1 \\ \text{right-hand limit of } f(x) = 0 \end{array}\right\} \text{average} = \frac{1+0}{2} = \frac{1}{2}$$

Furthermore, from (7.19) and the above theorem, we have

$$\underbrace{\int_0^\infty \frac{\sin k\cos(kx)}{k}dk}_{\text{Dirichlet's discontinuous factor}} = \begin{cases} \frac{\pi}{2} & 0 \le x < 1 \\ \frac{\pi}{4} & x = 1 \\ 0 & x > 1 \end{cases}$$

Dirichlet's discontinuous factor

Let us consider the case of $x = 0$ for the Dirichlet's discontinuous factor

$$\Rightarrow \int_0^\infty \frac{\sin k}{k}dk = \frac{\pi}{2} \qquad (7.20)$$

The integral in (7) is the limit of sine integral $Si(z)$ as $z \to \infty$

$$Si(z) \equiv \int_0^z \frac{\sin k}{k}dk \qquad (7.21)$$

The graph of $Si(z)$ is shown in the figure below (Fig. 241)

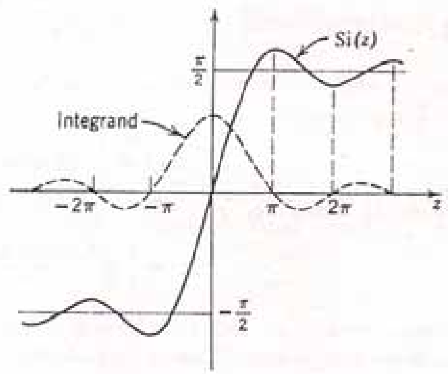


Fig. 241. Sine integral

Then a finite integral defined below will approach to $f(x)$ if $a \to \infty$.

$$\lim_{a\to\infty} \frac{2}{\pi}\int_0^a \frac{\sin k\cos(kx)}{k}dk \qquad (7.22) \;\to \frac{2}{\pi}\int_0^\infty \frac{\sin k\cos(kx)}{k}dk = f(x)$$

Fig.242 shows oscillations near the points of discontinuity of $f(x)$

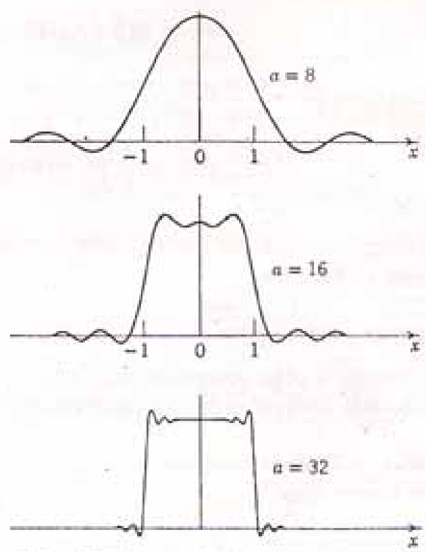


Fig. 242. The integral (9) for a = 8, 16 and 32

We might expect that these oscillations in Fig.242 disappear as a approaches infinity; however, with increasing a, these oscillations are shifted closer to the points $x = \pm 1$. This unexpected behavior is called the "Gibbs phenomenon" or "Gibbs oscillation". It can be explained by rewriting (9) in terms of the sine intergral as follows.

$$\frac{2}{\pi}\int_0^a \frac{\sin k \cos(kx)}{k} dk = \frac{1}{\pi}\int_0^a \frac{\sin(k+kx)}{k} dk + \frac{1}{\pi}\int_0^a \frac{\sin(k-kx)}{k} dk$$

$$\downarrow\ k + kx = t \qquad\qquad \downarrow\ k - kx = t$$

$$= \frac{1}{\pi}\int_0^{(x+1)a} \frac{\sin t}{t} dt - \frac{1}{\pi}\int_0^{(x-1)a} \frac{\sin t}{t} dt$$

$$\downarrow \quad Si(z) \equiv \int_0^z \frac{\sin k}{k} dk \qquad (7.23)$$

$$\frac{2}{\pi}\int_0^a \frac{\sin k \cos(kx)}{k} dk = \frac{1}{\pi} Si((x+1)a) - \frac{1}{\pi} Si((x-1)a)$$

Thus the oscillations in Fig.242 result from those in Fig.241. The increase of a equals to a transformation of the scale on the axis and cause the shift of the oscillations.

Ref: Navarra, A, W.F. Stern, and K. Miyakoda, 1994: Reduction of the Gibbs oscillation in spectral model simulation. *J. Climate*, **7**, 1169-1183.

- <u>Fourier</u> <u>Integrals</u> of <u>Even</u> and <u>Odd</u> Functions:

If $f(x)$ is an even function $\Rightarrow B(k)=0$ in (7.17)

and $$A(k)=2\int_0^\infty f(v)\cos kv dv \qquad (7.24)$$

So Eq.(5) or the Fourier integral of $f(x)$ reduces to a simpler form

$$f(x)=\frac{1}{\pi}\int_0^\infty A(k)\cos kx dk \qquad (f \text{ even}) \qquad (7.25)$$

Similarly if $f(x)$ is an odd function $\Rightarrow A(k)=0$ in (7.16)

And $$B(k)=2\int_0^\infty f(v)\sin kv dv \qquad (7.26)$$

Thus $$f(x)=\frac{1}{\pi}\int_0^\infty B(k)\sin kx dk \qquad (f \text{ odd}) \qquad (7.27)$$

Ex: Find the Fourier integral of $\begin{cases} f(x)=e^{-wx} \text{ and } w>0 & \text{when } x>0 \\ f(-x)=f(x) & \text{even function} \end{cases}$

Sol: sine $f(x)$ is even

(7.24) $\Rightarrow A(k)=2\int_0^\infty e^{-wv}\cos kv dv$

Using the integration by parts, we can get

$$\int e^{-wv}\cos kv dv=-\frac{w}{w^2+k^2}e^{-wv}\left(-\frac{k}{w}\sin kv+\cos kv\right)$$

When $v=0$, the RHS equals $-\frac{w}{w^2+k^2}$; when $v\to\infty$, it approaches zero because $e^{-\infty}\to 0$. Thus

$$A(k)=\frac{2w}{w^2+k^2}$$

Substituting this $A(k)$ in (7.25)

$$\Rightarrow f(x)=e^{-wx}=\frac{2w}{\pi}\int_0^\infty \frac{\cos kx}{w^2+k^2}dk \qquad (x>0, w>0)$$

Thus $$\int_0^\infty \frac{\cos kx}{w^2+k^2}dk=\frac{\pi}{2w}e^{-wx} \qquad (x>0, w>0) \qquad (7.26)$$

Similarly if $\begin{cases} f(x) = e^{-wx} \text{ and } w > 0 & \text{when } x > 0 \\ f(-x) = -f(x) & \text{odd function} \end{cases}$

By finding the Fourier integral, we will obtain this result

$$\int_0^\infty \frac{k \sin kx}{w^2 + k^2} dk = \frac{\pi}{2} e^{-wx} \qquad (x > 0, w > 0) \qquad (7.27)$$

Note that the integrals in (7.26) and (7.27) are the so-called "Laplace integrals"

- Complex form of the Fourier integral, or Fourier transform

Recall that

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty [A(k)\cos kx + B(k)\sin kx]dk \\ &= \frac{1}{\pi}\int_0^\infty \left[\left(\int_{-\infty}^\infty f(v)\cos kv dv\right)\cos kx + \left(\int_{-\infty}^\infty f(v)\sin kv dv\right)\sin kx\right]dk \\ &= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\underbrace{(\cos kv \cos kx + \sin kv \sin kx)}_{=\cos(kx - kv)}dv\right]dk \end{aligned}$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(kx - kv)dv\right]dk \qquad (7.28)$$

Since the integral from $-\infty$ to ∞ in (7.28) is an even function of *k*, Eq.(7.28) can be written

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\cos(kx - kv)dv\right]dk \qquad (7.29)$$

Given that

$$\frac{i}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)\sin(kx - kv)dv\right]dk = 0 \qquad (7.30)$$

Eq.(7.29) + Eq.(7.30) $\Rightarrow$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v)e^{ik(x-v)}dv\right]dk \qquad (7.31)$$

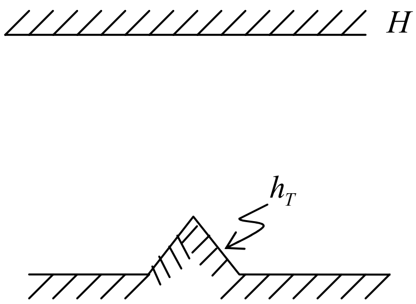
Eq.(7.31) is the complex form of the Fourier integral, and it can also be written as

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty c(k)e^{ikx}dk \quad \text{and} \quad c(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(v)e^{-ikv}dv \qquad (7.32)$$

where $f(x)$ is called the "Fourier transform" of $c(k)$, and $c(k)$ is called the "inverse Fourier transform" of $f(x)$.

Application of Fourier Integral for Problem in Atmospheric Science
—Force Topographic Rossby Waves

The simplest possible equation describing the topographic Rossby wave is the barotropic vorticity equation for a homogeneous fluid of variable depth.
[see Ch.4 of Holton's (1992) text book]



$$\frac{D_h}{Dt}(\varsigma + f) = (f + \varsigma)\frac{\partial w}{\partial z}$$

Quasi-geostrophic approximation

$\vec{\mathbf{V}} \sim \vec{\mathbf{V}}_{\mathbf{g}}$ and $|\varsigma_g| << f_0$

and $\frac{D_h}{Dt} = \frac{\partial}{\partial t} + \vec{\mathbf{V}} \cdot \nabla_h$

$$\frac{D_h}{Dt}(\varsigma_g + f) \approx f_0 \frac{\partial w}{\partial z}$$

Integrating the above equation from $z = h_T$ to $z = H$

Since $\varsigma_g \neq f(z), f \neq f(z) \Rightarrow$

$$\underbrace{(H - h_T)}_{\approx H}\left(\frac{\partial}{\partial t} + \vec{\mathbf{V}}_{\mathbf{g}} \cdot \nabla\right)(\varsigma_g + f) = f_0[w(H) - \underbrace{w(h_T)}_{\equiv \frac{Dh_T}{Dt}}]$$

$$\Rightarrow \quad H\left(\frac{\partial}{\partial t} + \vec{\mathbf{V}}_{\mathbf{g}} \cdot \nabla\right)(\varsigma_g + f) = -f_0 \frac{Dh_T}{Dt} \qquad (7.33)$$

Appling linearization and the mid-latitude β-plane approximation

$$\left(\begin{array}{c} u_g = \bar{u}_g + u'_g,\ v_g = v'_g,\ \varsigma_g = \varsigma'_g \\ \left(\frac{\partial}{\partial t} + \vec{\mathbf{V}}_{\mathbf{g}} \cdot \nabla\right)\varsigma_g \approx \frac{\partial \varsigma'_g}{\partial t} + \bar{u}_g \frac{\partial \varsigma'_g}{\partial t},\ \left(\frac{\partial}{\partial t} + \vec{\mathbf{V}}_{\mathbf{g}} \cdot \nabla\right) f \approx \beta v_g,\ \frac{Dh_T}{Dt} \approx \bar{u}_g \frac{\partial h_T}{\partial x} \end{array}\right)$$

$$(7.33) \ \Rightarrow \ \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\varsigma'_g + \beta v'_g = -\frac{f_0}{H}\bar{u}_g \frac{\partial h_T}{\partial x} \qquad (7.34)$$

Consider a special case of a sinusoidal lower boundary. We can specify the topography in this form (an infinite series of sinusoidal ridges)

$$h_T(x, y) = \mathrm{Re}\{h_0 e^{ikx}\}\cos ly$$

And the geostrophic wind and vorticity can be expressed by the perturbation stream function

$$\psi(x, y) = \mathrm{Re}\left[\psi_0 e^{-ikx}\right]\cos ly$$

Substituting $\psi(x, y)$ into (7.34) and assuming a steady state

$$\downarrow \qquad \varsigma'_g = \nabla^2\psi = -(k^2 + l^2)\psi, \quad v'_g = \frac{\partial \psi}{\partial x} = ik\psi, \quad \frac{\partial h_T}{\partial x} = ikh$$

We can find a steady-state solution $\psi(x, y)$ with complex amplitude ψ_0 as

$$\frac{f_0\psi_0}{g} = \frac{h_0}{\lambda_R^2\left(K^2 - K_s^2\right)} \qquad (7.35)$$

where $\lambda_R \equiv \frac{\sqrt{gH}}{f_0}$, the Rossby radius of deformation

$$K^2 = k^2 + l^2, \quad K_s^2 = \frac{\beta}{\bar{u}_g}$$

Thus, the streamfunction is either exactly in phase (ridges over mountains) or out of phase (troughs over mountains) with the topograph depending on the sign of $K^2 - K_s^2$. Note that in (23), if $K = K_s$, $\psi_0 \to \infty$. Thus this condition "$K^2 = K_s^2$" is the resonant condition for the stationary Rossby wave.

In reality, topographic feature tend to be isolated. Let us consider a simple bell-shape mountain

$$h(x) = \frac{h_m a^2}{a^2 + x^2}$$

where h_m is the maximum height and a is the half width of the mountain. This profile $h(x)$ can be represented by the Fourier transform

$$h(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{h}(k)e^{ikx}dk$$

$$\Rightarrow \hat{h}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{-ikx}dx$$

Similarly

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\psi}(k)e^{ikx}dk \Rightarrow \hat{\psi}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx$$

then $\nabla^2\psi \sim -k^2\hat{\psi}, \;\; \frac{\partial}{\partial x} \sim ik$

Substituting into (7.34)

$$\Rightarrow \;\; \frac{f_0\hat{\psi}}{g}=\frac{\hat{h}}{\lambda_R^2\left(k^2-k_s^2\right)}, \;\; \text{where } k_s^2 \equiv \frac{\beta}{\overline{u}_g}, \;\; \lambda_R^2=\frac{gH}{f_0} \qquad (7.36)$$

Using the inverse Fourier transform, we can find $\psi(x)$ solution back to the physical space (x)

$$\begin{aligned}\psi(x)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{\psi}(k)e^{ikx}dk \\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{g\hat{h}(k)}{f_0\lambda_R^2\left(k^2-k_s^2\right)}\right]e^{ikx}dk\end{aligned} \qquad (7.37)$$

For the wide mountain limit $(a \to \infty)$, the wavelength (L_x) of the maximum amplitude topographic Rossby wave is proportional to the mountain width,

If $a \to \infty \Rightarrow L_x \to \infty \Rightarrow k=\frac{2\pi}{L_x} \to 0 \Rightarrow k^2-k_s^2 \to -k_s^2$

(7.37) $\Rightarrow \psi \propto \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{h}(k)}{-k_s^2}e^{ikx}dk \propto \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{h}(k)e^{ikx}dk=-h(x)$

Thus the disturbance perturbation is negatively proportional to the mountain profile in the wide mountain limit.

On the other hand, for the narrow mountain limit ($a \to 0$),

$$\because \hat{\varsigma}_g(k)=\nabla^2\hat{\psi}=-k^2\hat{\psi} \Rightarrow \hat{\varsigma}_g=\frac{-k^2g\hat{h}}{f_0\lambda_R^2\left(k^2-k_s^2\right)}$$

Thus,

$$\varsigma_g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varsigma}_g(k) e^{ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-k^2 g\hat{h} e^{ikx}}{f_0 \lambda_R^2 \left(k^2 - k_s^2\right)} dk \propto \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{-\hat{h}(k)}{\left(1 - \frac{k_s^2}{k^2}\right)} e^{ikx} dk$$

As $a \to 0 \Rightarrow L_x \to 0 \Rightarrow k = \frac{2\pi}{L_x} \to \infty \Rightarrow \frac{k_s^2}{k^2} \to 0$

$\therefore \quad \varsigma_g \propto \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(k) e^{ikx} dk \propto -h(x)$

Then the disturbance perturbation is negatively proportional to the mountain profile in the narrow mountain limit.

Considering both wide and narrow mountain limits, we can conclude that the disturbance vorticity is negatively proportional to the mountain profile for the forced topographic Rossby waves.