Chapter 6 Power-Series Solution to the ODE

Recall a power series as

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a)+c_2(x-a)^2+\cdots \qquad (6.1)$$

with a radius of convergence R

$\Rightarrow$ this series converges to $f(x)$ for $|x-a|<R$

Ratio Test: $R=\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$ (6.2)

the radius of convergence for the series

Root Test: $R=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{|c_n|}}$ (6.3)

Note that if taking log in the limit, $\log\sqrt[n]{|c_n|}=\dfrac{1}{n}\log|c_n|$

Recall a Tayor-Series Expansion of $f(x)$ as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= f(x_0)+f'(x_0)(x-x_0)+\frac{f^{(2)}(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\cdots \end{aligned} \qquad (6.4)$$

Properties about the Power Series:

－added term by term

－multiplied term by term

－taking derivative term by term

－taking integral term by term

－Radius of Convergence is unaffected by differentiation and integration

Differencing a Power Series:

$$y(x)=\sum_{n=0}^{\infty} c_n (x-a)^n , \quad \text{where } y(a)=y_0,\ y'(a)=y'_0$$

$$\Rightarrow y'(x)=\sum_{n=0}^{\infty} c_n \cdot n(x-a)^{n-1} \quad (a \text{ is chosen by initial data point})$$

$$y''(x)=\sum_{n=0}^{\infty} c_n \cdot n \cdot (n-1)(x-a)^{n-2}$$

Note that you can rewrite $y'(x)$ and $y''(x)$ by shifting index of summation

For example, $y''(x)=\sum_{n=0}^{\infty} c_n \cdot n \cdot (n-1)(x-a)^{n-2}$

$\downarrow$ let $m=n-2$ or $n=m+2$

$$y''(x)=\sum_{m=-2}^{\infty} c_{m+2} \cdot (m+2)(m+1)(x-a)^m$$

$$=\sum_{m=0}^{\infty} c_{m+2} (m+2)(m+1)(x-a)^m$$

(since coefficient = 0 for the first two terms)

Ex: $\begin{cases} y''+4y=0 \\ y(0)=0,\ y'(0)=2 \end{cases}$,solve $y(x)=?$

Sol: Let $y(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ at $a=0$

$$\Rightarrow y(x)=\sum_{n=0}^{\infty} c_n x^n \Rightarrow y'(x)=\sum_{n=0}^{\infty} c_n \cdot n \cdot x^{n-1}$$

$$y''(x)=\sum_{n=0}^{\infty} c_n \cdot n \cdot (n-1) x^{n-2}$$

$$\therefore y''+4y=\left[\sum_{n=0}^{\infty} c_n \cdot n(n-1)x^{n-2} + 4\sum_{n=0}^{\infty} c_n x^n \right]=0$$

$$=\sum_{m=0}^{\infty} c_{m+2} (m+2)(m+1)x^m$$

$$=\sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1)x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} [c_{n+2} (n+2)(n+1)+4c_n]x^n=0$$

$\Rightarrow \quad c_{n+2}(n+2)(n+1)+4c_n = 0\,, \quad n = 0, 1, 2, 3, \cdots$ Recurrence Pormula

$n=0$: $c_2(2)(1)+4c_0=0 \;\Rightarrow c_2=-2c_0$

$n=1$: $c_3(3)(2)+4c_1=0 \;\Rightarrow c_3=-\dfrac{2}{3}c_1$

$n=2$: $c_4(4)(3)+4c_2=0 \;\Rightarrow c_4=-\dfrac{1}{3}c_2=-\dfrac{1}{3}(-2c_0)=\dfrac{2}{3}c_0$

$n=3$: $c_5(5)(4)+4c_3=0 \;\Rightarrow c_5=-\dfrac{1}{5}c_3=-\dfrac{1}{5}\left(-\dfrac{2}{3}c_1\right)=\dfrac{2}{15}c_1$

$\vdots$

all c_{2n} can be written in term of c_0, c_{2n+1} be eritten in terms of c_1

$$\Rightarrow \quad y(x)=\sum_{n=0}^{\infty}c_n x^n = c_0\left(1-2x^2+\frac{2}{3}x^4+\cdots\right)+c_1\left(x-\frac{2}{3}x^3+\frac{2}{15}x^5+\cdots\right)=c_o y_1+c_1 y_2$$

So y_1 and y_2 are the homogeneous solutions to this ODE

$\Rightarrow$ then decide c_1 and c_2 by ICs:

$y(0)=0=c_0$

and $y'(x)=c_0\left(-4x+\dfrac{8}{3}x^3+\cdots\right)+c_1\left(1-2x^2+\dfrac{2}{3}x^4+\cdots\right)$

$y'(0)=2=c_1$

$$\Rightarrow y(x)=2\left(x-\frac{2}{3}x^3+\frac{2}{15}x^5+\cdots\right)$$

Double Check the solution: $y''+4y=0$ & $y(0)=0$, $y'(0)=2$

$y''+4y=0 \;\Rightarrow y_1=\sin 2x,\; y_2=\cos 2x$

$\Rightarrow y_H=c_1\sin 2x+c_2\cos 2x$

$y_H'=2c_1\cos 2x-2c_2\sin 2x$

So, $y(0)=0=c_1\cdot 0+c_2\cdot 1 \quad \Rightarrow c_2=0$

$y'(0)=2=2c_1\cdot 1+2c_2\cdot 0 \quad \Rightarrow c_1=1$

$\therefore\; y_H=\sin 2x$

$$=(2x)-\frac{1}{3!}(2x)^3+\frac{1}{5!}(2x)^5-\frac{1}{7!}(2x)^7+-\cdots$$

$$=2\left(x-\frac{2}{3}x^3+\frac{2}{15}x^5+-\cdots\right)$$

Ex: $(2+t^2)y''-ty'-3y=0 \Rightarrow$ find $y(t)=?$

Sol: Assume a power-series solution as

$$y=\sum_{n=0}^{\infty} c_n t^n$$

$$\Rightarrow y'=\sum_{n=0}^{\infty} c_n \cdot n t^{n-1}$$

$$y''=\sum_{n=0}^{\infty} c_n \cdot n \cdot (n-1) t^{n-2}$$

$$\Rightarrow (2+t^2)\sum_{n=0}^{\infty} c_n n(n-1)t^{n-2} - t\sum_{n=0}^{\infty} c_n n t^{n-1} - 3\sum_{n=0}^{\infty} c_n t^n = 0$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} 2c_n n(n-1)t^{n-2}}_{\text{Let } n\text{-}2=m} + \sum_{n=0}^{\infty} c_n n(n-1)t^n - \sum_{n=0}^{\infty} c_n n t^n - \sum_{n=0}^{\infty} 3c_n t^n = 0$$

$$= \sum_{m=-2}^{\infty} 2c_{m+2}(m+2)(m+1)t^m$$

$$= \sum_{n=0}^{\infty} 2c_{n+2}(n+2)(n+1)t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[2c_{n+2}(n+2)(n+1)+c_n n(n-1)-c_n n-3c_n\right]t^n = 0$$

$$= \left[2(n+2)(n+1)c_{n+2}+(n-3)(n+1)c_n\right]$$

$$= (n+1)\left[2(n+2)c_{n+2}+(n-3)c_n\right] \quad \text{as} \quad n=0,1,2,\cdots$$

Recurrence formula

$$\Rightarrow \quad 2(n+2)c_{n+2}+(n-3)c_n = 0 \qquad \text{Note}: \lim_{n\to\infty}\left|\frac{c_n}{c_{n+2}}\right| = \lim_{n\to\infty}\left|\frac{2(n+2)}{n-3}\right| = 2$$

$$n=0: \ 2\cdot 2c_2+(-3)c_0 = 0 \Rightarrow c_2 = \frac{3}{4}c_0$$

$$n=1: \ 2\cdot 3c_3+(-2)c_1 = 0 \Rightarrow c_3 = \frac{1}{3}c_1$$

$$n=2: \ 2\cdot 4c_4+(-1)c_2 = 0 \Rightarrow c_4 = \frac{1}{8}c_2 = \frac{1}{8}\left(\frac{3}{4}\right)c_0 = \frac{2}{32}c_0$$

$$n=3: \ 2\cdot 5c_5+(0)c_3 = 0 \Rightarrow c_5 = 0$$

$$n=4: \ 2\cdot 6c_6+(1)c_4 = 0 \Rightarrow c_6 = -\frac{c_4}{12} = -\frac{1}{12}\left(\frac{3}{32}\right)c_0 = -\frac{1}{128}c_0$$

$$n=5: \ 2\cdot 7c_5+(2)c_5 = 0 \Rightarrow c_7 = \frac{c_5}{7} = 0, \ c_9, c_{11}, \cdots, = 0$$

$$\therefore \ y(t) = c_0\left(1+\frac{3}{4}t^2+\frac{3}{32}t^4-\frac{1}{128}t^6+\cdots\right)+c_1\left(t+\frac{1}{3}t^3\right)$$

Ex: $y'' + 4y = 0 \Rightarrow$ find $y(x) = ?$

Sol: Try the Power-Series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} c_n \cdot n x^{n-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n \cdot n(n-1) x^{n-2} = \sum_{m=-2}^{\infty} c_{m+2}(m+2)(m+1)x^m$$

$$= \sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n$$

Thus $y'' + 4y = \sum_{n=0}^{\infty} \left[c_{n+2}(n+2)(n+1) + 4c_n\right]x^n = 0$

Recurrence formula $\Rightarrow c_{n+2}(n+2)(n+1) + 4c_n = 0$, $n = 0, 1, 2, \cdots$

If $y'' + 4y = f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (Taylor-Series expansion)

then the corresponding recurrence formula is

$$\Rightarrow c_{n+2}(n+2)(n+1) + 4c_n = \frac{1}{n!}, \quad n = 0, 1, 2, \cdots$$

Note that we can take the Taylor-Series Expansion about $f(x)$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

this Power-Series method still works as solving the ODE !

- <u>Euler's Equation</u>:

ODE: $x^2 y''(x) + Axy'(x) + By(x) = 0$ (6.5) (Equidimensional ODE)

Let $y(x) = x^r$, then $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$\Rightarrow x^r\left[r(r-1) + Ar + B\right] = 0$$

$$\Rightarrow r^2 + (A-1)r + B = 0$$

$$\Rightarrow r_{\pm} = \frac{-(A-1)}{2} \pm \frac{\sqrt{(A-1)^2 - 4B}}{2} \qquad (6.6)$$

If $(A-1)^2 > 4B$: 2 distinct roots r_+ and r_-

$$\Rightarrow y(x) = c_1 x^{r_+} + c_2 x^{r_-} \qquad (6.7)$$

If $(A-1)^2 < 4B$: 2 distinct complex roots r_+ and r_-

let $r_{\pm} = c \pm iD$, where $c = \dfrac{-(A-1)}{2}, D = \dfrac{\sqrt{4B-(A-1)^2}}{2}$

$$\Rightarrow x^{r_+} = x^{c+iD} = x^c e^{iD\ln|x|}$$

$$x^{r_-} = x^{c-iD} = x^c e^{-iD\ln|x|}$$

$$\because e^{iD\ln|x|} = \cos(D\ln|x|) + i\sin(D\ln|x|)$$

$$\Rightarrow \begin{cases} y_1 = x^c \cos(D\ln|x|) \\ y_2 = x^c \sin(D\ln|x|) \\ \text{and } y = c_1 y_1 + c_2 y_2 \end{cases} \qquad (6.8)$$

If $(A-1)^2 = 4B$: repeated root for r

$$y_1 = x^{\frac{1-A}{2}} \quad , y_2 = y_1 \ln x \qquad (6.9)$$

Using the method of reduction of order to solve y_2

Ex: $xy'' + (1-x)y' + 2y = 0 \quad \Rightarrow$ find $y(x) = ?$

Sol: Try the power-series solution as

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow y'(x) = \sum_{n=0}^{\infty} c_n \cdot nx^{n-1}, \; y''(x) = \sum_{n=0}^{\infty} c_n \cdot n(n-1)x^{n-2}$$

Substitution into ODE $\Rightarrow x\sum_{n=0}^{\infty} c_n \cdot n(n-1)x^{n-2} + (1-x)\sum_{n=0}^{\infty} c_n \cdot nx^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \cdot n(n-1)x^{n-1} + \sum_{n=0}^{\infty} c_n \cdot nx^{n-1} - \sum_{n=0}^{\infty} c_n \cdot nx^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \cdot n^2 x^{n-1} + \sum_{n=0}^{\infty} c_n (2-n)x^n = 0$$

$$= \sum_{m=-1}^{\infty} c_{m+1}(m+1)^2 x^m$$

$$= \sum_{n=0}^{\infty} c_{n+1}(n+1)^2 x^n$$

$\Rightarrow$ Recurrence formula : $c_{n+1}(n+1)^2 + c_n(2-n) = 0, \; n = 0, 1, \; 2, \; \cdots$

$n=0:\ c_1\cdot 1^2+c_0(2-0)=0 \Rightarrow c_1=-2c_0$

$n=1:\ c_2\cdot 2^2+c_1(2-1)=0 \Rightarrow c_2=-\frac{1}{4}c_1=-\frac{1}{4}(-2c_0)=\frac{c_0}{2}$

$n=2:\ c_3\cdot 3^2+c_2(2-2)=0 \Rightarrow c_3=0$

$n=3:\ c_4\cdot 4^2+c_3(2-3)=0 \Rightarrow c_4=\frac{c_3}{16}=0$

$n=4:\ c_5\cdot 5^2+c_4(2-4)=0 \Rightarrow c_5=\frac{2c_4}{25}=0 \quad ,c_6=c_7=\cdots=0$

We get

$$y(x)=c_0\left(1-2x+\frac{x^2}{2}\right), \qquad \text{only 1 solution}$$
$$\equiv y_1(x)$$

Recheck the original ODE:

$$xy''+(1-x)y'+2y=0$$
$$\Rightarrow y''+\underbrace{\frac{1-x}{x}}y'+\underbrace{\frac{2}{x}}y=0$$

Singular point at x=0 !

Thus when trying the power-series solution about x=0, only one or even none solution can be found.

Using the method of "reduction of order" to find another solution $y_2(x)$

Let $y_2(x)=v(x)y_1(x)$ (6.10)

For ODE in this form

$$y''(x)+\underbrace{a(x)}_{=\frac{1-x}{x}}y'(x)+\underbrace{b(x)}_{=\frac{2}{x}}y(x)=0 \quad (6.11) \quad \leftrightarrow y''+\frac{1-x}{x}y'+\frac{2}{x}y=0$$

Substitution of (6.10) into (6.11) and y_1 a solution of this ODE

$$\Rightarrow v''(x)+\left(\frac{2y_1'(x)}{y_1(x)}+a(x)\right)v'(x)=0 \qquad (6.12)$$

Then $v'(x)$ can be found as

$$v'(x)=u(x)=\frac{1}{[y_1(x)]^2}\exp\left[-\int_0^x a(\tilde{x})d\tilde{x}\right]$$

$$\Rightarrow \begin{cases} v(x)=\int_0^x u(x^*)dx^*=\int_0^x \frac{1}{y_1^2(x^*)}\exp\left[-\int_0^{x^*} a(\tilde{x})d\tilde{x}\right]dx^* \\ \text{and } y_2(x)=y_1(x)v(x) \end{cases} \qquad (6.13)$$

But $\because\ y_1(x)=c_0\left(1-2x+\frac{x^2}{2}\right)\Rightarrow \frac{1}{y_1^2(x)}$: division over long power series

$$a(x)=\frac{1-x}{x},\quad \exp\left[-\int_0^{x^*}\frac{1-\tilde{x}}{\tilde{x}}d\tilde{x}\right]=e^{-\int_0^{x^*}\left(\frac{1}{x}-1\right)d\tilde{x}}=e^{-\ln x^*+x^*}=\frac{1}{x^*}e^{x^*}\rightarrow \text{long series.}$$

$\therefore\quad y_2(x)\rightarrow$ long power - series solution about $x=0$

and it still has singular point at $x=0$

Note that for the Euler's Equation: $y''+\frac{A(x)}{x}y'+\frac{B(x)}{x^2}y=0$

To find the solution $y(x)$,

First take Taylor series of $A(x)$ and $B(x)$

Then try the power-series solution as

$$y(x)=x^r\sum_{n=0}^{\infty}c_nx^n \qquad \Leftarrow \text{ Forbenius method}$$

to avoid the singular point at x=0

For this ODE: $y''(x)+a(x)y'(x)+b(x)y(x)=0$

<u>Ordinary point</u> $\Rightarrow$ if $a(x),b(x)$ have no singularities at this point x_0

then
$$a(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$
$$b(x)=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

<u>Regular singular point</u> $\Rightarrow$ if x_0 is a regular singular point to $a(x),\ b(x)$

then
$$(x-x_0)a(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$
$$(x-x_0)^2b(x)=\sum_{n=0}^{\infty}b_n(x-x_0)^2$$

e.g., take $x_0=0$, and

$$a(x)=\frac{a_0}{x}+a_1+a_2x+\cdots \qquad \Rightarrow xa(x)=a_0+a_1x+a_2x^2$$
$$b(x)=\frac{b_0}{x^2}+\frac{b_1}{x}+b_2+b_3x+\cdots \qquad \Rightarrow x^2b(x)=b_0+b_1x+b_2x^2+\cdots$$

Note that if $x=0$ is a regular singular point
and if $x=1$ is also a regular singular point
$\Rightarrow$ the radius of convergence around $x=0$ is 1

If expanding the power-series solution about an ordinary point
$\Rightarrow$ the power-series solution will work !
and we will get two solutions as $y_1=\sum_{n=0}^{\infty} c_n x^n, y_2=\sum_{n=0}^{\infty} d_n x^n$

If expanding the power-series solution about regular singular point
$\Rightarrow$ we will get only one power-series solution
$\Rightarrow$ should try the Frobenius method,
i.e., Assume two solutions as $y_1=x^r\sum_{n=0}^{\infty} c_n x^n, y_2=x^r\sum_{n=0}^{\infty} d_n x^n$

For the ODE:

$$y''+a(x)y'+b(x)y=0 \;, \qquad (6.14)$$

where $a(x)=\dfrac{1}{x}\sum_{n=0}^{\infty} a_n x^n, b(x)=\dfrac{1}{x^2}\sum_{n=0}^{\infty} b_n x^n$

To solve this ODE, let us use the Frobenius method

$$y=x^r\sum_{n=0}^{\infty} c_n x^n=\sum_{n=0}^{\infty} c_n x^{n+r} \qquad (6.15)$$

$$\Rightarrow y'=\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$$

$$\Rightarrow y''=\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$$

Substituting into the ODE, i.e., Eq. (6.14),

$$y''+a(x)y'+b(x)y=\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}+\frac{1}{x}\sum_{n=0}^{\infty} a_n x^n\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$$

$$+\frac{1}{x^2}\sum_{n=0}^{\infty} b_n x^n\sum_{n=0}^{\infty} c_n x^{n+r}=0$$

Consider the lowest power term:

$$\Rightarrow c_0 r(r-1)x^{r-2}+c_0 r\left(\frac{a_0}{x}+a_1+\cdots\right)x^{r-1}+c_0\left(\frac{b_0}{x^2}+\frac{b_1}{x}+b_2+\cdots\right)x^r$$

$$\Rightarrow c_0[r(r-1)+ra_0+b_0]x^{r-2} + \text{high - order terms of } x^{r-1}, x^r, \cdots = 0$$

$$\because y(x) = x^r\sum_{n=0}^{\infty} c_n x^n, \text{ so assume } c_0 \neq 0$$

$$\Rightarrow r(r-1)+ra_0+b_0 = 0$$

or

$$r^2 + r(a_0-1) + b_0 = 0 \qquad (6.16) \qquad \text{Indicial Equations}$$

There are two roots (r_1, r_2) in Eq. (6.16). Let us consider three cases.

Three cases for r_1 and r_2:

(1) $r_1 - r_2$ is not an integer
(2) $r_1 = r_2$ repeated root
(3) $r_1 - r_2$ is an integer

Case Ⅰ): $r_1 - r_2$ is not an integer

Assume two solutions as

$$\begin{cases} y_1(x) = x^{r_1}\sum_{n=0}^{\infty} c_n x^n \\ y_2(x) = x^{r_2}\sum_{n=0}^{\infty} c_n x^n \end{cases} \qquad (6.17)$$

Then we will have two linearly independent solutions for Eq. (6.14)

Ex: $4xy'' + 2y' + y = 0 \Rightarrow$ find $y(x) = ?$

Sol: $y'' + \dfrac{1}{2x}y' + \dfrac{1}{4x}y = 0 \ \Rightarrow a(x) = \dfrac{1}{2x},\ b(x) = \dfrac{1}{4x}$

$$(x-0)a(x) = \frac{1}{2} \Rightarrow a_0 = \frac{1}{2}$$

$$(x-0)^2 b(x) = \frac{1}{4}x \Rightarrow b_0 = 0$$

$\therefore$ Indicial Equations $\Rightarrow r^2 + \left(\dfrac{1}{2} - 1\right)r + 0 = 0$

$$\Rightarrow r^2 - \frac{1}{2}r = 0,\ r_1 = 0,\ \ r_2 = \frac{1}{2}$$

$$r_1 = \frac{1}{2} \Rightarrow y_1 = x^{\frac{1}{2}}\sum_{n=0}^{\infty} c_n x^n$$

$$r_2 = 0 \Rightarrow y_2 = \sum_{n=0}^{\infty} d_n x^n$$

To find $y_1(x)$: $\because\ y_1 = x^{\frac{1}{2}}\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}}$

$$\Rightarrow y_1' = \sum_{n=0}^{\infty} c_n\left(n+\frac{1}{2}\right)x^{n-\frac{1}{2}}$$

$$\Rightarrow y_1'' = \sum_{n=0}^{\infty} c_n\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)x^{n-\frac{3}{2}}$$

Substituting y_1 into the ODE $4xy'' + 2y' + y = 0$

$$\Rightarrow 4x\sum_{n=0}^{\infty} c_n\left(n^2-\frac{1}{4}\right)x^{n-\frac{3}{2}} + 2\sum_{n=0}^{\infty} c_n\left(n+\frac{1}{2}\right)x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 4c_n\left(n^2-\frac{1}{4}\right)x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} c_n(2n+1)x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n\left[\left(4n^2-1\right)+(2n+1)\right]x^{n-\frac{1}{2}} + \sum_{n=0}^{\infty} c_n x^{n+\frac{1}{2}} = 0$$

$\downarrow$ $4n^2 + 2n = 2n(2n+1)$ $\qquad$ $\downarrow$ let $n = k-1$

$$\sum_{n=0}^{\infty} c_n \cdot 2n(n+1)x^{n-\frac{1}{2}} \qquad\qquad \sum_{k=1}^{\infty} c_{k-1}x^{k-\frac{1}{2}} = \sum_{n=1}^{\infty} c_{n-1}x^{n-\frac{1}{2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \cdot 2n(2n+1)x^{n-\frac{1}{2}} + \sum_{n=1}^{\infty} c_{n-1}x^{n-\frac{1}{2}} = 0$$

$$\Rightarrow c_0 \cdot 2\cdot 0\cdot 1\cdot x^{-\frac{1}{2}} + \sum_{n=1}^{\infty}\left[2n(2n+1)c_n + c_{n-1}\right]x^{n-\frac{1}{2}} = 0$$

=0 (as n = 0)

$\Rightarrow$ Recurrence formula: $2n(2n+1)c_n + c_{n-1} = 0, \quad n = 1, 2, \cdots$

$n = 1$: $2\cdot 1\cdot 3c_1 + c_0 = 0 \Rightarrow \mathrm{c}_1 = -\frac{1}{6}c_0$

$n = 2$: $2\cdot 2\cdot 5c_2 + c_1 = 0 \Rightarrow \mathrm{c}_2 = -\frac{1}{20}c_1 = -\frac{1}{20}\left(-\frac{c_0}{6}\right) = \frac{1}{120}c_0$

$n = 3$: $2\cdot 3\cdot 7c_3 + c_2 = 0 \Rightarrow \mathrm{c}_3 = -\frac{1}{42}c_2 = -\frac{1}{42}\left(\frac{c_0}{120}\right) = -\frac{1}{5040}c_0$

$\vdots$

$$\therefore y_1(x) = c_0 x^{\frac{1}{2}}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \frac{1}{5040}x^3 + \cdots\right)$$

$$= c_0 x^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)!!} x^n$$

(Note that $5!! = 5\cdot 3\cdot 1$, $7!! = 7\cdot 5\cdot 3\cdot 1$)

and one can use the same method to find $y_2(x) = \sum_{n=0}^{\infty} d_n x^n$

Case Ⅱ): $r_1 = r_2$ repeated root

Ex: $y'' + \frac{1}{x}y' + \frac{2}{x}y = 0$ (*) $\Rightarrow$ find $y(x) = ?$

Sol: $\begin{cases} a(x) = \frac{1}{x} \Rightarrow & a_0 = 1 \\ b(x) = \frac{2}{x} \Rightarrow b_0 = 0, & b_1 = 2 \end{cases}$

So the indicial equation to the ODE is

$$\begin{aligned} & r^2 + (a_0 - 1)r + b_0 = 0 \\ \Rightarrow & r^2 + (1-1)r + 0 = 0 \\ \Rightarrow & r_1 = r_2 = 0 \end{aligned}$$

$\therefore y_1(x) = x^0 \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^n$, we need to find $y_2(x)$

Substituting the power-series solution of $y_1(x)$ into the ODE (*)

We can get (the details is skipped)

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n$$

Using the "reduction-of-order" method to get $y_2(x)$ i.e.,

Let $y_2(x) = y_1(x)v(x)$

Substituting $y_2(x)$ into (*) and given that $y_1(x)$ is a solution to (*)

$$\Rightarrow v''(x) + \left(2\frac{y_1'}{y_1} + a(x)\right)v'(x) = 0$$

$$\therefore v'(x) = \frac{1}{y_1^2(x)} \exp\left[-\int_0^x a(\tilde{x})\, d\tilde{x}\right]$$

$$\downarrow \quad \text{since } a(\tilde{x}) = \frac{1}{\tilde{x}} \text{ in this problem}$$

$$v'(x) = \frac{1}{y_1^2(x)} \exp\left[-\int_0^x \frac{1}{\tilde{x}}\, d\tilde{x}\right] = \frac{1}{y_1^2(x)} \underbrace{\exp[-\ln x]}_{= e^{-\ln x} = \frac{1}{x}}$$

Thus $\quad v'(x) = \dfrac{1}{x y_1^2(x)}$ and $y_1(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} x^n$

$$\Rightarrow \quad v(x) = \int^x \frac{d\bar{x}}{\bar{x}\left[\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(n!)^2} \bar{x}^n\right]^2}$$

$$= \int^x \frac{d\bar{x}}{\bar{x}} \left\{b_0 + b_1 \bar{x} + b_2 \bar{x}^2 + \cdots\right\}$$

$$= b_0 \ln x + b_1 x + b_2 \frac{x^2}{2} + b_3 \frac{x^3}{3} + \cdots$$

$$\therefore y_2(x) = y_1(x) v(x)$$

$$= y_1(x)\left[b_0 \ln x + b_1 x + \frac{b_2}{2} x^2 + \frac{b_3}{3} x^3 + \cdots\right]$$

$$= b_0 (\ln x) y_1(x) + \underbrace{\underbrace{\left(b_1 x + \frac{b_2}{2} x^2 + \cdots\right)}_{\text{series}} \underbrace{y_1(x)}_{\text{series}}}_{\text{power series}}$$

Thus for the ODE: $\quad y'' + a(x) y' + b(x) y = 0$

where $\quad a(x) = \dfrac{a_0}{x} + a_1 + \cdots, \quad b(x) = \dfrac{b_0}{x^2} + \dfrac{b_1}{x} + b_2 + \cdots$

If assume $\quad y(x) = x^r \displaystyle\sum_{n=0}^{\infty} c_n x^n$

two roots for r in the indicial equation:

$$r^2 + (a_0 - 1) r + b_0 = 0 \Rightarrow r_1, r_2$$

If $\quad r_1 = r_2 \Rightarrow$

$$
\begin{aligned}
y_1(x) &= x^{r_1}\sum_{n=0}^{\infty} c_n x^n \\
y_2(x) &= \ln x \cdot y_1(x) + x^{r_1}\sum_{n=0}^{\infty} d_n x^n
\end{aligned}
\tag{6.18}
$$

Further discussion on the repeated-root case:

Indicial Equation is $r^2 + (a_0 - 1)r + b_0 = 0$

If r_1 is a repeated root $\Rightarrow$ $r_1 = \dfrac{1-a_0}{2}$

$$\therefore y_1 = x^{\frac{1-a_0}{2}}\sum_{n=0}^{\infty} c_n x^n$$

Then by the reduction of order,

$$
\begin{aligned}
y_2(x) &= v(x)y_1(x) \\
\Rightarrow v'(x) &= \frac{1}{y_1^2(x)}\exp\left[-\int^x a(\tilde{x})\,d\tilde{x}\right] \\
&= \frac{1}{\left[x^{\frac{1-a_0}{2}}\sum_{n=0}^{\infty} c_n x^n\right]^2}\exp\left[-\int^x\left(\frac{a_0}{\tilde{x}} + a_1 + a_2\tilde{x} + \cdots\right)d\tilde{x}\right] \\
&= \frac{1}{x^{1-a_0}\left[\sum_{n=0}^{\infty} c_n x^n\right]^2}\exp\left[-a_0\ln x - a_1 x - \frac{a_2}{2}x^2 - \cdots\right]
\end{aligned}
$$

$$\frac{1}{x^{1-a_0}}\sum_{n=0}^{\infty} c_n^* x^n \ (\because c_0^* \neq 0),\ \exp[\ ] = \frac{1}{x^{a_0}}\sum_{n=0}^{\infty} b_n x^n \ \ (\because b_0 \neq 0)$$

$$v'(x) = \frac{1}{x}\sum_{n=0}^{\infty} b_n^* x^n \qquad (b_0^* \neq 0)$$

$$\therefore v(x) = b_0^*\ln x + b_1^* x + \frac{b_2^*}{2}x^2 + \cdots$$

$$y_2(x) = v(x)y_1(x) = \left(b_0^*\ln x + b_1^* x + \cdots\right)x^{r_1}\sum_{n=0}^{\infty} c_n x^n$$

$$\Rightarrow y_2(x) = y_1(x)\ln x + \underbrace{\left(b_1^* x + \frac{b_2^*}{2}x^2 + \cdots\right)x^{r_1}\sum_{n=0}^{\infty} c_n x^n}$$

$$= x^{r_1}\underbrace{\underbrace{\left(b_1^* x + \frac{b_2^*}{2}x^2 + \cdots\right)}_{\text{series}}\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\text{series}}}_{=\text{series}}$$

$$= x^{r_1}\sum_{n=0}^{\infty} d_n x^n$$

$$\downarrow$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=0}^{\infty} d_n x^n \qquad \text{(A)}$$

Note that let $\dfrac{1}{1 - x + x^2 - x^3 + - \cdots} = c_0 + c_1 x + c_2 x^2 + \cdots$ (B)

$$\Rightarrow 1 = \left(1 - x + x^2 - x^3 + - \cdots\right)\left(c_0 + c_1 x + c_2 x^2 + \cdots\right)$$

$$= c_0 + (c_1 - c_0)x + (c_2 - c_1 + c_0)x^2 + (\cdots)x^3 + \cdots$$

$\Rightarrow c_0 = 1$

$c_1 - c_0 = 0 \Rightarrow c_1 = c_0 = 1$

$c_2 - c_1 + c_0 = 0 \Rightarrow c_2 = c_1 - c_0 = 0$

$\because$ just try $x = 0$ in (B) $\Rightarrow$ $c_0 = 1$

$\therefore$ we know that there is no negative power in the RHS of Eq. (B)

Thus, for $f(x) = \dfrac{1}{1 - x + x^2 - x^3 + - \cdots}$, no singularity at $x = 0$

Also from (A) $\Rightarrow$

$$y_2' = y_1' \ln x + y_1 \frac{1}{x} + \sum_{n=0}^{\infty} d_n (n + r_1) x^{n + r_1 - 1}$$

and

$$y_2'' = y_1'' \ln x + \frac{2}{x}y_1' - \frac{1}{x^2}y_1 + \sum_{n=0}^{\infty} d_n (n + r_1)(n + r_1 - 1)x^{n + r_1 - 2}$$

Substituting into the ODE: $y'' + a(x)y' + b(x)y = 0$

$$\Rightarrow \left[\underline{\underline{y_1''\ln x}} + \frac{2}{x}y_1' - \frac{1}{x^2}y_1 + \sum_{n=0}^{\infty} d_n (n + r_1)(n + r_1 - 1)x^{n + r_1 - 2}\right]$$

$$+ a(x)\left[\underline{\underline{y_1' \ln x}} + \frac{1}{x}y_1 + \sum_{n=0}^{\infty} d_n (n + r_1)x^{n + r_1 - 1}\right] + b(x)\left[\underline{\underline{y_1 \ln x}} + x^{r_1}\sum_{n=0}^{\infty} d_n x^n\right] = 0$$

$\underline{\underline{\cdots\cdots}} = \ln x\left(y_1'' + a(x)y_1' + b(x)y_1\right) = 0$ $\because$ y_1 is the solution to the ODE

$y'' + a(x)y + b(x)y = 0$

$$\downarrow$$

$$\sum_{n=0}^{\infty} d_n (n+r_1)(n+r_1-1)x^{n+r_1-2} + a(x)\sum_{n=0}^{\infty} d_n (n+r_1)x^{n+r_1-1} + b(x)\sum_{n=0}^{\infty} d_n x^{n+r_1}$$
$$= -\frac{2}{x} y_1' + \frac{1}{x^2} y_1 - a(x)\frac{1}{x} y_1 \qquad \text{(C)}$$

$\because$ the solution $y_1 = x^{r_1}\sum_{n=0}^{\infty} c_n x^n$ is known

$\therefore$ RHS term of Eq.(C) are known

If we further take the Taylor-Series expansion of $a(x)$ and $b(x)$ and then compare the coefficients in Eq. (C) terms by terms.

$\Rightarrow$ We can find some recurrence formula like this

$$f_1(n)d_n + f_2(n)d_{n-1} + f_3(n)d_{n-2} = g(n), \quad n = 1, 2, 3, \cdots$$

Case III): $r_1 - r_2$ is an integer

The solution is sometimes like Case Ⅰ), sometimes like Case Ⅱ)

If $r_1 > r_2$: let $y_1(x) = x^{r_1}\sum_{n=0}^{\infty} c_n x^n$

$$\text{and} \begin{cases} y_2(x) = x^{r_2}\sum_{n=0}^{\infty} d_n x^n \\ \text{OR} \\ y_2(x) = \ln x \cdot y_1(x) + x^{r_2}\sum_{n=0}^{\infty} d_n x^n \end{cases} \qquad (6.19)$$

Bessel's Equation of order ν :

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \qquad (6.20)$$

$$\frac{\text{Eq.(6.20)}}{x^2} \Rightarrow y'' + \underbrace{\frac{1}{x}}_{=a(x)} y' + \underbrace{\left(1 - \frac{\nu^2}{x^2}\right)}_{=b(x)} y = 0 \qquad (6.21)$$

$$\Rightarrow a_0 = 1, \quad b_0 = -\nu^2$$

So the indicial equation for the ODE in (6.21) is

$$r^2 + (a_0 - 1)r + b_0 = 0$$
$$\Rightarrow r^2 - \nu^2 = 0 \quad \Rightarrow r = \pm\nu$$

Try $\nu = 1$: $r_1 = 1,\ r_2 = -1 \Rightarrow r_1 - r_2 = 2$

The first solution $\Rightarrow$

$$y_1(x) = x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1}$$
$$\Rightarrow y_1'(x) = \sum_{n=0}^{\infty} c_n (n+1) x^n$$
$$\Rightarrow y_1''(x) = \sum_{n=0}^{\infty} c_n (n+1) c x^{n-1}$$

Substituting into (6.20)

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$
$$\therefore \sum_{n=0}^{\infty} c_n (n+1) n x^{n+1} + \sum_{n=0}^{\infty} c_n (n+1) x^{n+1} + (x^2 - 1)\sum_{n=0}^{\infty} c_n x^{n+1} = 0$$
$$\Rightarrow \sum_{n=0}^{\infty} c_n \underbrace{[(n+1)n + (n+1) - 1]}_{\substack{(n+1)^2 - 1 \\ = n^2 + 2n}} x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+3} = 0$$

$\downarrow$ let $n = m - 2$

$$= \sum_{m=2}^{\infty} c_{m-2} x^{m+1} = \sum_{n=2}^{\infty} c_{n-2} x^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \cdot n(n+2) x^{n+1} + \sum_{n=2}^{\infty} c_{n-2} x^{n+1} = 0$$

$n = 0$: $c_0 \cdot 0 \cdot 2 \cdot x^1 = 0 \Rightarrow c_0 \neq 0$ (can be anything)

$n = 1$: $c_1 \cdot 1 \cdot 3 \cdot x^2 = 0 \Rightarrow c_1 = 0$

$n \geq 2$: we can get the recurrence formula as

$$n(n+2)c_n + c_{n-2} = 0 \qquad n = 2, 3, 4, \cdots$$
$$\Rightarrow c_n = -\frac{c_{n-2}}{n(n+2)}$$

$$\therefore c_1 = c_3 = c_5 = c_7 = \cdots = 0$$

$$c_2 = -\frac{c_0}{2\cdot 4},\ c_4 = -\frac{c_2}{4\cdot 6} = c\left(\frac{1}{4\cdot 6}\right)\left(-\frac{c_0}{2\cdot 4}\right) = \frac{c_2}{2\cdot 4\cdot 4\cdot 6}$$

$$c_6 = -\frac{c_4}{6\cdot 8} = -\left(\frac{1}{6\cdot 8}\right)\left(\frac{c_0}{2\cdot 4\cdot 6}\right) = -\frac{c_0}{6\cdot 2\cdot 4\cdot 6\cdot 8}, \cdots$$

$$\Rightarrow \quad y_1(x) = x\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}\cdot n!(n+1)!} \quad , \text{ is called } J_1(x)$$

$$J_1(x) \equiv \left(\frac{x}{2}\right)\sum_{n=0}^{x}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n}}{n!(n+1)!}$$

Bessel's function of the 1st kind of order 1

Note that

$$J_\nu(x) = \left|\frac{x}{2}\right|^\nu \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n}}{n!\Gamma(n+1+\nu)} \qquad (6.22)$$

where $\Gamma(m+1) = m!$ if m is an integer

Bessel's function of the 1st kind of order ν

<u>Find the 2nd solution</u> for $r_2 = -1$:

$$y_2(x) = x^{-1}\sum_{n=0}^{\infty} d_n x^n$$

$$\Rightarrow y_2'(x) = \sum_{n=0}^{\infty} d_n (n-1) x^{n-2}$$

$$y_2''(x) = \sum_{n=0}^{\infty} d_n (n-1)(n-2) x^{n-3}$$

Substituting into the ODE below:

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} d_n (n-1)(n-2) x^{n-1} + \sum_{n=0}^{\infty} d_n (n-1) x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} - \sum_{n=0}^{\infty} d_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} d_n \left[n^2 - 3n + 2 + n - 1 - 1\right] x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} = 0$$

$$= n^2 - 2n = n(n-2)$$

$$\Rightarrow \sum_{n=0}^{\infty} d_n n(n-2) x^{n-1} + \sum_{n=0}^{\infty} d_n x^{n+1} = 0$$

$x^{-1}(n=0)$: $d_0 \cdot 0 \cdot (-2) = 0 \Rightarrow d_0 =$ anything (not necessarily zero)

$x^0(n=1)$: $d_0 \cdot 1 \cdot (-1) = 0 \Rightarrow d_1 = 1$ contradiction !

$x\ (n=2)$: $d_2 \cdot 2 \cdot (0) + d_0 = 0 \Rightarrow d_0 = 0$

Thus $y_2(x) = \ln x \cdot y_1(x) + x^{-1}\sum_{n=0}^{\infty} d_n x^n$

(Note that this $y_2(x)$ has singularity at $x = 0$)

Consider the Bessel's equation of order ν:

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0\,, \quad \nu \text{ is not an integer}$$

Its corresponding Bessel's solution of the first kind is

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!\,\Gamma(n+\nu+1)}$$

and $$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!\,\Gamma(n-\nu+1)} \qquad (6.23)$$

(a second linearly independent solution)

Thus the general solution to this Bessel's equation of order ν is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x) \quad (\text{for } \nu \neq \text{integer}) \qquad (6.24)$$

Note that in practice, we don't expand out $J_\nu(x)$ and $J_{-\nu}(x)$,

$\Rightarrow$ we just use it like $\sin x$ and $\cos x$ (knowing their properties) to represent a particular solution.

For the Bessel's equation of order ν (but ν is an integer):

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0 \qquad \nu \text{ is an integer}$$

Its 2 linearly independent solutions are

$$\begin{cases} y_1(x) = J_\nu(x) \\ y_2(x) = \ln x \cdot J_\nu(x) + x^{-\nu} \sum_{n=0}^{\infty} d_n x^n \end{cases} \qquad (6.25)$$

And the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Given that $Y_\nu(x) =$ Bessel's function of the second kind of order ν

If ν is not an integer,

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}\left[J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)\right] \qquad (6.26)$$

If ν is an integer *n*,

$$Y_n(x) = \lim_{\nu \to n} \frac{1}{\sin(\nu\pi)} \left[J_\nu(x) \cos(\nu\pi) - J_{-\nu}(x) \right] \qquad (6.27)$$

Because $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent, then the general solution to the Bessel's equation of order ν can be written as

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x) \qquad (6.28)$$

Note that

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - x\sin\theta) d\theta, \quad n \text{ is an integer}$$

Properties of Bessel's Functions:

(1) $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$

(2) $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$

(3) $xJ_p'(x) = xJ_{p-1}(x) - pJ_p(x)$

(4) $xJ_p'(x) = pJ_p(x) - xJ_{p+1}(x)$

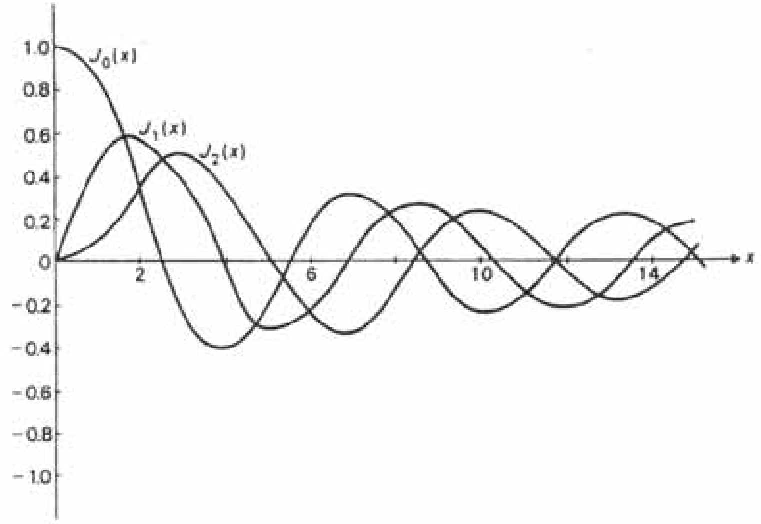
(5) $xJ_{p+1}(x) - 2pJ_p(x) + xJ_{p-1}(x) = 0$

(6) $2J_p'(x) = J_{p-1}(x) - J_{p+1}(x)$

Theorem 2 A general solution of Bessel's equation of order p is

$$y(x) = AJ_p(x) + BY_p(x), \quad x \neq 0.$$

Graphs of the functions Y_0, Y_1, and Y_2 are given in Fig. 1.



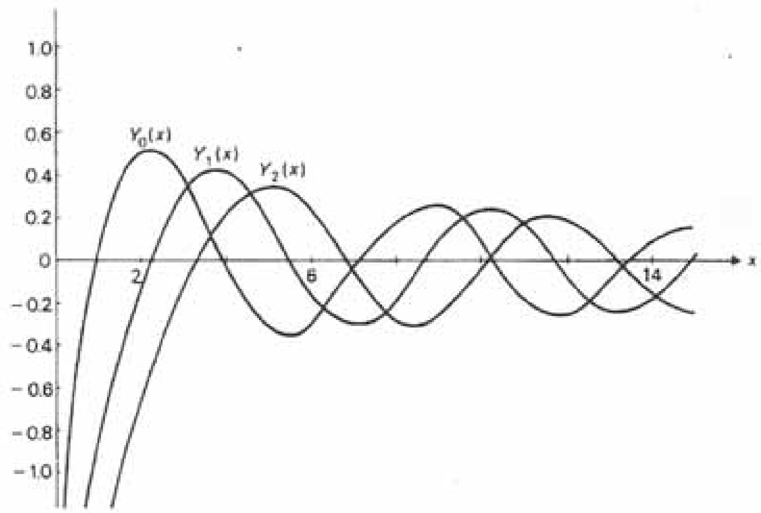


Figure 1

Comparison Table 1:

ODE	$y'' + \omega^2 y = 0$	$y'' - \omega^2 y = 0$
Solution	$y_1 = \sin(\omega x)$ $y_2 = \cos(\omega x)$	$y_1 = e^{\omega x}$ or $\sinh(\omega x)$ $y_1 = e^{-\omega x}$ or $\cosh(\omega x)$

Comparison Table 2:

ODE	$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$	$x^2 y'' + xy' + (x^2 + \nu^2)y = 0$
Solution	$y_1 = J_\nu(x)$ $y_2 = Y_\nu(x)$	$y_1 = I_\nu(x)$ $y_2 = K_\nu(x)$

$$\nabla^2\Phi = \begin{cases} 0 & \text{Lapalace Equation (or Potential Eq.)} \\ \dfrac{\partial\Phi}{\partial t} & \text{Diffusion Equation} \\ \dfrac{\partial^2\Phi}{\partial t^2} & \text{Wave Equation} \end{cases}$$

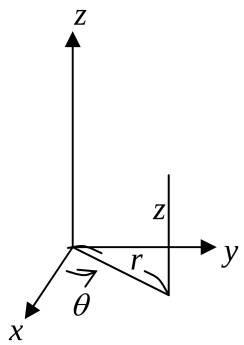
Assume cylindrical symmetry (i.e., no θ dependence)
then the Laplace Equation can be written as

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial t^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (6.30)$$

By separation of variable $\Phi(r,z) = R(r)Z(z)$ (because of no θ dependence)

$$\Rightarrow \frac{\partial^2\Phi}{\partial r^2} = R''(r)Z(z)$$

$$\frac{\partial\Phi}{\partial r} = R'(r)Z(z)$$



$$\frac{\partial^2 \Phi}{\partial z^2} = R(r) Z''(z)$$

Substituting into the Laplace Equation and dividing by RZ

$$\frac{1}{RZ}\left\{R''Z + \frac{1}{r}R'Z + RZ'' = 0\right\}$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{\frac{1}{r}R'(r)}{R(r)} = -\frac{Z''(z)}{Z(z)} = \text{constant} = -\lambda$$

$\because$ both sides are functions of different independent variables (r, z)

$$\Rightarrow \begin{cases} R'' + \dfrac{1}{r}R' = -\lambda R & (6.31) \\ Z'' - \lambda Z = 0 \Rightarrow Z(z) = c_1 e^{\sqrt{\lambda}\, z} + c_2 e^{-\sqrt{\lambda}\, z} & (6.32) \end{cases}$$

For the $R(r)$ equation

$$r^2 \times \left\{R''(r) + \frac{1}{r}R'(r) + \lambda R(r) = 0\right\}$$

$$\Rightarrow r^2 R''(r) + rR'(r) + \lambda r^2 R(r) = 0$$

$\updownarrow$ versus

$$x^2 y'' + xy' + \left(x^2 - 0^2\right)y = 0$$ It is similar to the Bessel's equation of order 0 !

Ex: $y'' + xy = 0$ (A)

Objective: try to make the above ODE similar to Bessel's type ODE

Sol:

Let $y(x) = x^\alpha u(z)$, $z = cx^\beta$ $\quad$ [Note in here, $u = f(x)$ only]

$$= x^\alpha u\left(cx^\beta\right)$$

$$\Rightarrow y'(x) = \alpha\, x^{\alpha-1} u(Z(x)) + x^\alpha u'(z) \underline{\underline{z'(x)}} \quad \text{, here } z'(x) = c\beta x^{\beta-1}$$

$$y''(x) = \alpha(\alpha-1)x^{\alpha-2}u(z) + \alpha\, x^{\alpha-1}u'(z)\underline{\underline{z'(x)}} + c\beta(\alpha+\beta-1)x^{\alpha+\beta-2}u'(z)$$
$$+ c\beta x^{\alpha+\beta-1}u''(z)\underline{\underline{z'(x)}}$$

$c\beta x^{\beta-1}$

OR

$$y''(x) = c^2\beta^2 x^{\alpha+2\beta-2}u''(z) + (2\alpha+\beta-1)c\beta x^{\alpha+\beta-2}u'(z) + \alpha(\alpha-1)x^{\alpha-2}u(z)$$

Substituting into the ODE in Eq.(A):

$$c^2\beta^2 x^{\alpha+2\beta-2}u''(z)+(2\alpha+\beta-1)c\beta x^{\alpha+\beta-2}u'(z)+\left[\alpha(\alpha-1)x^{\alpha-2}+x\cdot x^{\alpha}\right]u(z)=0$$

$$u''(z)+\frac{(2\alpha+\beta-1)c\beta x^{\beta-2}}{c^2\beta^2x^{2\beta-2}}u'(z)+\left[\frac{\alpha(\alpha-1)x^{-2}}{c^2\beta^2x^{2\beta-2}}+\frac{x}{c^2\beta^2x^{2\beta-2}}\right]u(z)=0$$

OR $$u''(z)+\left(\frac{2\alpha+\beta-1}{c\beta}x^{-\beta}\right)u'(z)+\left(\frac{\alpha(\alpha-1)}{c^2\beta^2}x^{-2\beta}+\frac{1}{c^2\beta^2}x^{-2\beta+3}\right)u(z)=0 \qquad \text{(B)}$$

Recall that the Bessel's D.E. for *u(z)* is

$$z^2u''+zu+(z^2-\nu^2)u=0$$

$$\Rightarrow u''+\frac{1}{z}u+\left(1-\frac{\nu^2}{z^2}\right)u=0 \qquad \text{(C)}$$

Comparing Eq.(C) with Eq.(B) and Noting that $z=cx^{\beta}$

Requiring $\frac{2\alpha+\beta-1}{c\beta}\left(\frac{c}{z}\right)=\frac{1}{z}\Rightarrow\alpha=\frac{1}{2}$

Thus Eq.(B) can be rewritten as

$$u''(z)+\frac{1}{z}u'(z)+\left[\frac{1}{c^2\beta^2}x^{-2\beta+3}-\frac{1}{4c^2\beta^2}x^{-2\beta}\right]u(z)=0 \qquad \text{(D)}$$

$$\because z=cx^{\beta}\ ,\therefore x=\left(\frac{z}{c}\right)^{\frac{1}{\beta}}$$

$$[\ ]=\frac{1}{c^2\beta^2}\left(\left(\frac{z}{c}\right)^{\frac{1}{\beta}}\right)^{-2\beta+3}-\frac{1}{4c^2\beta^2}\left(\left(\frac{z}{c}\right)^{\frac{1}{\beta}}\right)^{-2\beta}$$

$$=\frac{1}{c^2\beta^2}\left(\frac{z}{c}\right)^{\frac{3}{\beta}-2}-\frac{1}{4\beta^2}\frac{1}{z^2}$$

let it be "1" $\qquad$ let it be "$\frac{\nu^2}{z^2}$"

So $\frac{3}{\beta}-2=0\Rightarrow\beta=\frac{3}{2}$ $\quad\left[\because\left(\frac{z}{c}\right)^0=1\right]$

$$\frac{1}{4\beta^2}=\nu^2\Rightarrow\nu=\frac{1}{4\beta}=\frac{1}{2\left(\frac{3}{2}\right)}=\frac{1}{3}$$

$$\frac{1}{c^2\beta^2}=1\Rightarrow c=\frac{1}{\beta}=\frac{2}{3}$$

$$u''(z)+\frac{1}{z}u'(z)+\left[1-\frac{\left(\frac{1}{3}\right)^2}{z^2}\right]u(z)=0 \qquad \text{(E)}$$

Bessel's equation of order 1/3

With the transformations of $y(x)=x^{\frac{1}{2}}u(x),\ z=\frac{2}{3}x^{\frac{3}{2}}$

from the original ODE $y''+xy=0$

Then the general solution to Eq.(E) is

$$u(z)=c_1J_{\frac{1}{3}}(z)+c_2Y_{\frac{1}{3}}(z) \qquad \text{(F)}$$

Transferring back to the original ODE:

Since $z=cx^{\beta}=\frac{2}{3}x^{\frac{3}{2}}$

$$y(x)=x^{\alpha}u(z)=x^{\frac{1}{2}}u(z)$$

Thus for the ODE $y''+xy=0$

Its general solution can be written as

$$y(x)=c_1x^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)+c_2x^{\frac{1}{2}}Y_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) \qquad \text{(G)}$$

- Legendre Differential Equation:

$$\left(1-x^2\right)y''-2xy'+p(p+1)y=0, \quad \text{where } y=y(x)$$

$$\Rightarrow \left[\left(1-x^2\right)y'\right]'+p(p+1)y=0 \qquad (6.33)$$

Since " x = 0 "is not a singular point for the Legendre D.E.

We can use the power-series (polynomial) method to find the general solution

If $p=0 \Rightarrow 0^{\text{th}}$-order polynomial solution

$p=1 \Rightarrow 1^{\text{st}}$-order polynomial solution

$\vdots$

$p=n \Rightarrow$ nth-order Legendre polynomial

Define: $P_n(x)$ is a Legendre polynomial with order n which satisfies $P_n(1)=1$

Note that $P_n(x)$ can be described by the spherical harmonic $x=\cos\theta$,

where θ is the latitude

<u>More</u> about the <u>Spherical</u> <u>Harmonic</u>:

Ref: Holton (1992)'s text book, pp. 451–453
Consider the Barotropic Vorticity Equation in the spherical coordinate,

$$\frac{D}{Dt}(\zeta+2\Omega\sin\phi)=0 \qquad (1)$$

where $\zeta=\nabla^2\psi$, ψ is a streamfunction

$$\frac{D}{Dt}\equiv\frac{\partial}{\partial t}+\frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\phi} \qquad (2)$$

a is the Earth's radius, ϕ is the latitude, λ is the longitude
u is the Zonal wind, v is the meridional wind

Note that in (1), ζ is relative vorticity, and $2\Omega\sin\phi$ is planetary vorticity,
so Eq. (1) describes the conservation of absolute vorticity.

The continuity equation in spherical coordinate is

$$\frac{1}{a}\frac{\partial}{\partial\lambda}\left(\frac{u}{\cos\phi}\right)+\frac{1}{a}\frac{\partial}{\partial\mu}(v\cos\phi)=0 \qquad (3)$$

where $\mu\equiv\sin\phi$

i) Note that recall the horizontal velocity divergence in spherical coordinate is

$$\nabla\cdot\vec{\mathbf{V}}_{\mathbf{n}}=\frac{1}{a\cos\phi}\left[\frac{\partial v}{\partial\lambda}+\frac{\partial(v\cos\phi)}{\partial\phi}\right]$$

$$=\frac{1}{a\cos\phi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(v\cos\phi)$$ see Appendix B in Holton's (1992) book

$$=\frac{1}{a}\frac{\partial}{\partial\lambda}\left(\frac{u}{\cos\phi}\right)+\frac{1}{a}\frac{\partial}{\partial\sin\phi}(v\cos\phi)$$

ii) Note that the horizontal velocity divergence in Cartesian coordinate is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\xleftrightarrow[dy=ad\phi]{dx=a\cos\phi d\lambda}\frac{1}{a\cos\phi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(v\cos\phi)=0$$

Thus the streamfunction is related to zonal and meridional velocities by

$$\frac{u}{\cos\phi} = -\frac{1}{a}\frac{\partial\psi}{\partial\mu}\ , v\cos\phi = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} \qquad (4)$$

Then the vorticity equation can be expressed as

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{1}{a^2}\left[\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda}\left(\nabla^2\psi\right) - \frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu}\left(\nabla^2\psi\right)\right] - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} \qquad (5)$$

Derivation of (5):

$$(1)+(2) \Rightarrow \left(\frac{\partial}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\phi}\right)\left(\nabla^2\psi + 2\Omega\sin\phi\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\nabla^2\psi\right) = -\frac{u}{a\cos\phi}\frac{\partial}{\partial\lambda}\left(\nabla^2\psi\right) - \frac{v}{a}\frac{\partial}{\partial\phi}\left(\nabla^2\psi\right) - \frac{v}{a}2\Omega\cos\phi$$

$$\because\ u = -\frac{\cos\phi}{a}\frac{\partial\psi}{\partial\mu}, v = \frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda} \qquad (4)$$

$$= -\frac{1}{a\cos\phi}\left(-\frac{\cos\phi}{a}\frac{\partial\psi}{\partial\mu}\right)\frac{\partial}{\partial\lambda}\left(\nabla^2\psi\right) - \frac{1}{a}\left(\frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda}\right)\frac{\partial}{\partial\phi}\left(\nabla^2\psi\right)$$

$$-\frac{1}{a}\left(\frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda}\right)\left(2\Omega\cos\phi\right)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\nabla^2\psi\right) = \frac{1}{a^2}\frac{\partial\psi}{\partial\mu}\frac{\partial}{\partial\lambda}\left(\nabla^2\psi\right) - \frac{1}{a^2}\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\mu}\left(\nabla^2\psi\right) - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}$$

Note that the vorticity can be expressed in terms of derivative of μ and λ:

$$\nabla^2\psi = \frac{1}{a^2}\left\{\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\lambda^2}\right\} \qquad (6)$$

Derivation of (6): Recall from Appendix B in Holton's (1992) text book

$$\nabla_h^2\psi = \frac{1}{a^2\cos^2\phi}\left[\frac{\partial^2\psi}{\partial\lambda^2} + \cos\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\psi}{\partial\phi}\right)\right]$$

$$\because\ \mu \equiv \sin\phi,\ \cos^2\phi = 1-\mu^2$$

$$\frac{\partial}{\partial\mu} = \frac{\partial}{\partial\sin\phi} = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}$$

$$= \frac{1}{a^2\left(1-\mu^2\right)}\frac{\partial^2\psi}{\partial\lambda^2} + \frac{1}{a^2}\underbrace{\frac{\partial}{\cos\phi\,\partial\phi}}_{\partial\sin\phi}\left(\underbrace{\cos^2\phi}_{1-\mu^2}\underbrace{\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\phi}}_{\partial\sin\phi}\right)$$

$$= \frac{1}{a^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu}\right]$$

$$\nabla_h^2\psi = \frac{1}{a^2}\left\{\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\psi}{\partial\mu}\right] + \frac{1}{1-\mu^2}\frac{\partial^2\psi}{\partial\lambda^2}\right\}$$

The appropriate orthogonal basis function to Eq.(6) are the "spherical harmonics", which are defined as

$$Y_r(\mu,\lambda) \equiv P_r(\mu)e^{im\lambda} \qquad (7)$$

where $\gamma \equiv (n,m)$ is a vector containing the integer indices for the spherical harmonics. These are given by $m = 0, \pm 1, \pm 2, \pm 3, \cdots, n = 1, 2, 3, \cdots$, where it is required that $|m| \le n$. Here, P_r designates an associated Legendre function of the first kind of degree *n*. From (7) it is clear that *m* designates the zonal wave number. It can be shown that $n-|m|$ designates the number of nodes of P_r in the interval $-1 < \mu < 1$ (i.e., between the south and north poles) and thus measures the meridional scales of the spherical harmonics. The structures of a few spherical harmonic are shown in the following figure from Holton's text book.

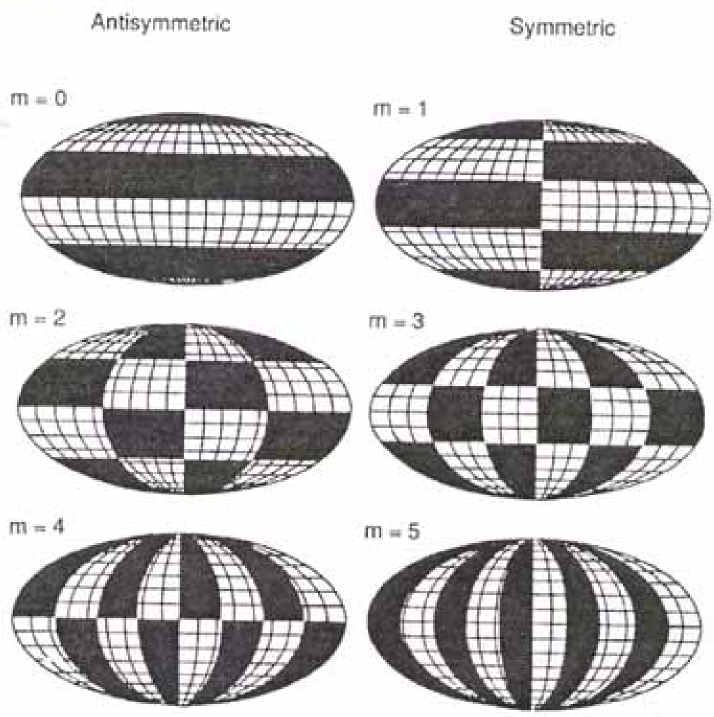


Fig. 13.3 Patterns of positive and negative regions for the spherical harmonic functions with $n = 5$ and $m = 0, 1, 2, 3, 4, 5$. (After Washington and Parkinson, 1986, adapted from Baer, 1972.)

An important property of the spherical harmonic is that they satisfy this relationship

$$\nabla^2 Y_r = -\frac{n(n+1)}{a^2} Y_r \qquad (8)$$

Thus the Laplacian of a spherical harmonic (Y_r) is proportional to the function itself. This implies that the vorticity associated with a particular spherical harmonic component is simply proportional to the streamfunction for the same component.

For the spectral solution on the sphere, the streamfunction can be expanded in a finite series of spherical harmonics by

$$\psi(\lambda, \mu, t) = \sum_{\gamma} \psi_{\gamma}(t) Y_{\gamma}(\mu, \lambda) \qquad (9)$$

Where ψ_r is the complex amplitude for the Y_r spherical harmonic and the summation is over both *n* and *m*.

The individual spherical harmonic coefficients ψ_r are related to the streamfunction $\psi(\lambda, \mu)$ through the inverse transform

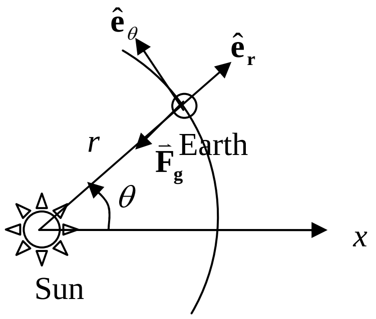
$$\psi_r(t) = \frac{1}{4\pi} \int_S Y_{\gamma}^{*} \psi(\lambda, \mu, t) dS \qquad (10)$$

where $dS = d\mu \, d\lambda$

Y_{γ}^{*} = the complex conjugates of Y_r

An example of nonlinear ODE: Earth's rotation around Sun

Ex:



Consider the planetary motion of Earth
(Centripetal force exerted by Sun)

$$\vec{\mathbf{F}}_\mathbf{g} = m\vec{\mathbf{a}}$$

$$-G\frac{Mm}{r^2}\hat{\mathbf{e}}_\mathbf{r} = m\left[\left(\ddot{r} - r\dot{\theta}\right)\hat{\mathbf{e}}_\mathbf{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\mathbf{e}}_\theta\right]$$

(Note that $\dot{\theta} = \dfrac{d\theta}{dt}$, $\dot{r} = \dfrac{dr}{dt}$)

$\hat{\mathbf{e}}_\mathbf{r}$ component: $\quad -\dfrac{GM}{r^2} = \ddot{r} - r\dot{\theta}^2 \qquad$ (A)

$\hat{\mathbf{e}}_\theta$ component: $\quad 0 = 2\dot{r}\dot{\theta} + r\ddot{\theta} \qquad$ (B)

$$r \times (\text{B}) \Rightarrow 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$\therefore \quad r^2\dot{\theta} = \text{constant} = h \quad$ (C) $\quad$ i.e., angular momentum conservation of earth

Then $\dot{\theta} = \dfrac{h}{r^2} \qquad$ (D)

Substituting (D) into (A) and let $GM = k$

$$\Rightarrow -\frac{k}{r^2} = \ddot{r} - r\left(\frac{h}{r^2}\right)^2 = \ddot{r} - \frac{h^2}{r^3}$$

$\Rightarrow \ddot{r} - \dfrac{h^2}{r^3} + \dfrac{k}{r^2} = 0 \qquad$ (E) $\quad$ Still a nonlinear ODE for *r(t)*

Apply the energy integral to Eq.(E)

i.e., $\dot{r} \times (\text{E}) \Rightarrow \dot{r}\,\ddot{r} + \dot{r}\left(\dfrac{k}{r^2} - \dfrac{h^2}{r^3}\right) = 0$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}\dot{r}^2 + \left(\frac{h^2}{2r^2} - \frac{k}{r}\right)\right] = 0$$

$$\Rightarrow \frac{1}{2}\dot{r} + \left(\frac{h^2}{2r^2} - \frac{k}{r}\right) = E = \text{constant} \qquad \text{(F)}$$

Note that in Eq. (F), $\frac{1}{2}\dot{r}$ is kinetic energy, $\frac{h^2}{2r^2}$ is the energy associated with angular rotation, and $-\frac{k}{r}$ is potential energy. Thus, Eq. (F) describes total energy conservation for the dynamical systems.

Eq. (F) can be rewritten as $\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = E - \frac{h^2}{2r^2} + \frac{k}{r}$

$$\Rightarrow \frac{dr}{dt} = \sqrt{2E - \frac{h^2}{r^2} + \frac{2k}{r}} \qquad \text{(G)}$$

Note that Eq. (G) is a separable D.E., so it can be solved in principle.

A simple and elegant way to solve Eq. (G) is introduced as follows:

Let $r = \frac{1}{u}$, and consider r, u as functions of θ, rather than function of t

$$\Rightarrow u(\theta) = \frac{1}{r} \qquad \text{(H)}$$

$$\Rightarrow u'(\theta) = -\frac{1}{r^2}\dot{r}\frac{dt}{d\theta}$$

$$\frac{dr}{dt}\frac{dt}{d\theta} = \frac{dr}{d\theta}$$

but $\frac{dt}{d\theta} = \frac{1}{\dot{\theta}}$ since $\dot{\theta} = \frac{d\theta}{dt}$

$$u'(\theta) = -\frac{\dot{r}}{r^2\theta}$$

But $r^2\dot{\theta} = h = \text{constant}$ (C)

$$\Rightarrow u'(\theta) = -\frac{\dot{r}}{h} \qquad \text{(I)}$$

and $u''(\theta) = -\frac{1}{h}\frac{d}{dt}(\dot{r})\frac{dt}{d\theta} = -\frac{\ddot{r}}{h\dot{\theta}} = -\frac{\ddot{r}\,r^2}{h^2}$ (J)

Substituting (H), (I),(J) into (E) and using that $r = \frac{1}{u}$,

$$\ddot{r} - \frac{h^2}{r^3} + \frac{k}{r^2} = 0$$

$$\Rightarrow -\frac{h^2}{r^2} u''(\theta) - h^2 u^3(\theta) + k u^2(\theta) = 0$$

$$h^2 u^2$$

$$\Rightarrow \quad u''(\theta) + u(\theta) - \frac{k}{h^2} = 0 \qquad \text{(K)}$$

where $k = GM$, $h = r^2 \dot{\theta}$

So the solution to ODE in Eq.(K) is

$$u(\theta) = \frac{k}{h^2} + A\cos(\theta - \phi)$$

$$\therefore r(\theta) = \frac{1}{u(\theta)} = \frac{1}{\frac{k}{h^2} + A\cos(\theta - \phi)}$$

OR

$$r(\theta) = \frac{\frac{h^2}{k}}{1 + \frac{Ah^2}{k}\cos(\theta - \phi)}$$

Thus

$$r(\theta) = \frac{l}{1 + e\cos(\theta - \phi)} \qquad \text{(L)}$$

Note that $r(\theta)$ is an equation for ellipse, $e = \frac{Ah^2}{k}$ is the eccentricity, and $l = \frac{h^2}{k}$ is the semi-major axis. Hence Eq. (L) for $r[\theta(t)]$ represents the track of Earth rotating around the Sun in the polar coordinate !