Chapter 5 Ordinary Differential Equations

Definition:

Differential Equation $\Rightarrow$ one or more equations involve certain known quantities (independent variables) and certain unknown quantities (dependent variables) and their derivatives.

Classifying D.E.:

derivatives $\Rightarrow$ derivatives in just one variable $\Rightarrow$ O. D.E. (Ordinary Diff. Eq.)

Two or more independent variables $\Rightarrow$ Partial D.E.

e.g. $\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0$

ODE's:

Order of equations $\Rightarrow$ highest derivative present

e.g., $y'''(x) + \sin x \cdot y = 0 \rightarrow 3^{\text{rd}}$ order ODE

$(y')^3 + \cos x \cdot y = 0 \rightarrow 1^{\text{rd}}$ order ODE (nonlinear)

Initial Value Problem (I.V.P.)

Boundary Value Problem (B.V. P.)

e.g., $F\left(x, y, y', y'', \cdots, y^{(n)}(x)\right) = 0$

I.V.P. $\underbrace{y(a) = y_0, y'(a) = y_1, \cdots, y^{(n-1)}(a) = y_{n-1}}$

n^{th} order ODE with n ICs (initial conditions)

B.V.P.: An example of boundary value problems

$$F(x, y, y', y'') = 0$$

BCs: $\begin{cases} y(a) = y_0 \\ y'(b) = y_1 \\ y''(c) = y_2 \end{cases}$, BCs specified at 3 different points

1^{st} order O.D.E. :

A general form $F(x, y, y') = 0$

Some types of 1^{st} order O.D.E. :

1) Separable Equation 2) Exact Equation

3) Substitution Technique 4). Linear 1^{st} order Eq.

1) Separable Equation:

$$y'(x) = f(x, y) \qquad (5.1)$$

If $f(x, y) = \dfrac{g(x)}{h(y)}$, then Eq.(5.1) is called "separable"

i.e., $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)} \Rightarrow \quad h(y)dy = g(x)dx$

$$\Rightarrow \quad H(y) = G(x) + c$$

Furtherly if $H(y)$ has the inverse function H^{-1}

$$\Rightarrow \ y = H^{-1}(G(x) + c)$$

Note a common mostake : $H(y) = G(x) + c$

$$\Rightarrow y = H^{-1}(G(x)) + c$$

Constant c must be inside the inverse function !

Ex : $y' = 2xy$

$$\frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{y} = 2xdx$$

$$\Rightarrow \ln y = x^2 + c_1$$

$$\Rightarrow y = e^{x^2 + c_1} = \underbrace{e^{c_1}}_{\equiv c} e^{x^2} = ce^{x^2} \neq e^{x^2} + c$$

Ex : $y'(x) = g(x)$ if it is integrable

$$\Rightarrow y(x) = \int g(x)dx + c = G(x) + c$$

Ex : $y''(x) = \sin x$ can be integrated twice

$$\Rightarrow y'(x) = -\cos x + c_1$$

$$\Rightarrow y(x) = -\sin x + c_1 x + c_2$$

General Form : $y^{(n)}(x) = f(x)$

$$\Rightarrow y(x) = \underbrace{\int\int \cdots \int}_{\text{n times}} f(x)dx + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0$$

2) Exact Solution:

Ex : $$\frac{dy}{dx} = \frac{xy - 2x + y - 2}{\frac{y}{x} + y - 4 - \frac{4}{x}} = \frac{x(y-2) + (y-2)}{y\left(\frac{1}{x}+1\right) - 4\left(\frac{1}{x}+1\right)} = \frac{(x+1)(y-2)}{(y-4)\left(\frac{1}{x}+1\right)}$$

$$= \frac{\cancel{(x+1)}(y-2)}{(y-4)\frac{1}{x}\cancel{(1+x)}} = \frac{x(y-2)}{y-4}$$

$$\Rightarrow \int \frac{y-4}{y-2} dy = \int x dx + c$$

$$\Rightarrow \int \left[1 - \frac{2}{y-2}\right] dy = \int x dx + c$$

$$\Rightarrow y - 2\ln(y-2) = \frac{x^2}{2} + c$$

3) Substitution Techniques:

$$y'(x) = f(x, y), \text{ where } f(x, y) = F\left(\frac{y}{x}\right)$$

then we can use the substitution

$$y'(x) = F\left(\frac{y}{x}\right) \qquad (+)$$

Let $z = \frac{y}{x}$, $y = zx$, and $y'(x) = z + \underbrace{xz'}_{}$

$\because z = z(x)$

Thus Eq. (+) becomes

$\Rightarrow xz' + z = F(z)$ the resulting equation is separable

$$\Rightarrow x\frac{dz}{dx} = F(z) - z$$

$$\Rightarrow \int \frac{dz}{F(z) - z} = \int \frac{dx}{x}$$

Assume $\int \frac{dz}{F(z) - z} = \phi(z)$, then

$$\Rightarrow \phi(z) = \ln x + c$$

$$\Rightarrow z = \phi^{-1}[\ln x + c] \quad \text{and } z = \frac{y}{x}$$

$$\Rightarrow y = x\phi^{-1}[\ln x + c]$$

Ex: $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2} = \left(\dfrac{y}{x}\right)^2 + 2\left(\dfrac{y}{x}\right)$, find $y(x) = ?$

Sol: Let $z = \dfrac{y}{x}$, and $y = zx$

$$\Rightarrow \frac{dy}{dx} = x\frac{dz}{dx} + z = z^2 + 2z$$

$$\Rightarrow x\frac{dz}{dx} = z^2 + z$$

$$\Rightarrow \underbrace{\int \frac{dz}{z^2 + z}}_{\text{LHS}} = \underbrace{\int \frac{dx}{x}}_{\text{RHS}}$$

$$\therefore \text{ LHS} = \int \frac{dz}{z(z+1)} = \int \left[\frac{1}{z} - \frac{1}{z+1}\right] dz = \ln z - \ln(z+1) = \ln\left(\frac{z}{z+1}\right)$$

$$\text{RHS} = \ln x + c$$

$$\Rightarrow \ln\left(\frac{z}{z+1}\right) = \ln x + c = \ln(c_0 x)$$

$$\equiv \ln c_0 \text{ (assumed)}$$

$$\Rightarrow \frac{\text{z}}{\text{z}+1} = c_0 x \Rightarrow z = (z+1)c_0 x$$

$$\Rightarrow z(1 - c_0 x) = c_0 x \Rightarrow z = \frac{c_0 x}{1 - c_0 x} = \frac{y}{x}$$

$$\Rightarrow y = \frac{c_0 x^2}{1 - c_0 x}$$

Let substitute the solution back into ODE to check its satisfication of Eq (+) ?

Check: $$\frac{dy}{dx} = \frac{(1 - c_0 x)(2c_0 x) - c_0 x^2(-c_0)}{(1 - c_0 x)^2} = \frac{c_0 x(2 - c_0 x)}{(1 - c_0 x)^2}$$

$$\frac{y^2 + 2xy}{x^2} = \frac{\dfrac{c_0{}^2 x^4}{(1 - c_0 x)^2} + 2x\dfrac{c_0 x^2}{1 - c_0 x}}{x^2}$$

the same!

$$= \frac{c_0{}^2 x^2}{(1 - c_0 x)^2} + \frac{2c_0 x}{1 - c_0 x} = \frac{c_0{}^2 x^2 + 2c_0 x(1 - c_0 x)}{(1 - C_0 x)^2}$$

$$= \frac{c_0 x[c_0 x + 2(1 - c_0 x)]}{(1 - c_0 x)^2} = \frac{c_0 x(2 - c_0 x)}{(1 - c_0 x)^2}$$

4) 1^{st} order Linear Equations :

$$a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$

$$\Rightarrow \quad \frac{dy}{dx} + a(x)y = f(x) \quad (5.2) \quad \text{, where } a(x) = \frac{a_0(x)}{a_1(x)},\ f(x) = \frac{b(x)}{a_1(x)}$$

By using the integration factor $\mu(x)$

Eq.(5.2) $\cdot\, \mu(x) \Rightarrow \quad \mu(x)y' + \mu(x)a(x)y = \mu(x)f(x) \qquad$ (A)

LHS of Eq.(A) is assumed to be "exact differential"

Because $\qquad \frac{d}{dx}(\mu(x)y) = \mu(x)y' + \mu'(x)y \qquad$ (B)

Substituting Eq.(B) into Eq.(A), then LHS of Eq.(5.2) is "exact differential"

only if $\quad \mu'(x) = \mu(x)a(x) \qquad$ (C)

$$\Rightarrow \frac{d\mu(x)}{dx} = \mu(x)a(x) \quad \text{or} \quad \int \frac{d\mu}{\mu} = \int a(x)\,dx$$

$$\Rightarrow \ln\mu = \int_0^x a(\tilde{x})\,d\tilde{x}$$

$$\Rightarrow \mu(x) = \exp\left[\int_0^x a(\tilde{x})\,d\tilde{x}\right] \qquad \text{(D)}$$

Then Eq.(A) is integrable

$$\frac{d}{dx}(\mu(x)\,y(x)) = \mu(x)\,f(x)$$

$$\Rightarrow \mu(x)\,y(x) = \int \mu(x)\,f(x)\,dx + c$$

$$\Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,f(x)\,dx + c\right] \text{, where } \mu(x) = e^{\int_0^x a(\tilde{x})\,d\tilde{x}} \qquad (5.3)$$

Ex: $y' + \frac{3}{x} y = \frac{e^x}{x^2}$, find $y(x) = ?$

Sol : Choose $a(x) = \frac{3}{x}$

then $\mu(x) = \exp\left(\int \frac{3}{x}\, dx\right) = \exp(3\ln x) = x^3$

ODE : $\mu(x) y' + \frac{3}{x}\mu\, y = \mu \frac{e^x}{x^2} = \frac{d}{dx}(\mu\, y)$

$$\Rightarrow \frac{d}{dx}\left(x^3 y\right) = x^3 \frac{e^x}{x^2} \qquad (\because \mu = x^3)$$

$$xe^x$$

$$\Rightarrow x^3 y = \int xe^x dx + c = (x-1)e^x + c \quad \text{(Using the integration by parts)}$$

$$\Rightarrow y = \frac{x-1}{x^3} e^x + \frac{c}{x^3}$$

If further consider the initial condition (IC) such as

$$y(1) = 2$$

$$\Rightarrow y(1) = \frac{1-1}{1} e^0 + \frac{c}{1} = c = 2 \Rightarrow y = \frac{x-1}{x^3} e^x + \frac{2}{x^3}$$

- Combination Technique (Substitution→Linear ODE)

Bernoulli's Differential Eq.: $$\frac{dy}{dx} + a(x) y = f(x) y^n \qquad (5.4)$$

Substitution technique: assume $z(x) = (y(x))^{1-n}$

$$\frac{dz}{dx} = (1-n)(y(x))^{-n} \frac{dy}{dx}$$

Eq. (5.4) $\times (1-n)(y(x))^{-n} \Rightarrow$

$$\underbrace{(1-n) y^{-n} \frac{dy}{dx}}_{= \frac{dz}{dx}} + a(x)\underbrace{(1-n) y^{1-n}}_{= a(x)(1-n) z(x)} = f(x)(1-n) \qquad (A)$$

$$\Rightarrow \quad \frac{dz}{dx} + a(x)(1-n) z(x) = f(x)(1-n) \qquad (B)$$

This ODE is linear in z

$\Rightarrow$ One can solve (B) for $z(x)$ (i.e., assume solvable)

and $z = y^{1-n}$

$$\Rightarrow \quad y(x) = z^{\frac{1}{1-n}}$$

Recall Type 2) Exact Equation:

$$M(x,y)dx + N(x,y)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} \quad (5.5)$$

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then this ODE is "exact"

Question: How does this condition for "exact" ODE come from?

Let $g(x,y) = c$, then take total differential

$$\Rightarrow dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$

then $M = \dfrac{\partial g}{\partial x}$, $N = \dfrac{\partial g}{\partial y}$

$$\because \frac{\partial M}{\partial y} = \frac{\partial^2 g}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 g}{\partial x \partial y}$$

they are the same if the 2D derivative exists

$\therefore$ "$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$" is the condition of the ODE $Mdx + Mdy = 0$ is "exact"

Note that $\dfrac{\partial g}{\partial x} = M$, $\dfrac{\partial g}{\partial y} = N$ $\Rightarrow$ $g(x,y) = \int_0^x M(\tilde{x}, y)d\tilde{x} + h(y)$, $g(x,y) = \int_0^y N(x,\tilde{y})d\tilde{y} + k(x)$

Ex: $\underbrace{(y\cos x)}_{M}dx + \underbrace{(\sin x + 2y)}_{N}dy = 0 \Rightarrow$ find the corresponding $g(x,y) = ?$

Sol: $\dfrac{\partial M}{\partial y} = \cos x$, $\dfrac{\partial N}{\partial x} = \cos x \Rightarrow$ the ODE is exact

$$\therefore\ g(x,y) = \int_0^x y\cos\tilde{x}\, d\tilde{x} + h(y) = c$$

$$= y\sin x + h(y) = c$$

And $\dfrac{\partial g}{\partial y} = \sin x + h'(y) = N = \sin x + 2y$

$$\Rightarrow\ h'(y) = 2y, \quad h(y) = y^2$$

$\therefore\ g(x,y) = y\sin x + y^2 =$ constant ← implicit formula for *y(x)*

Another way :

$$g(x,y)=\int_0^y[\sin x+2\tilde{y}]d\tilde{y}+k(x)=c$$

$$=y\sin x+y^2+k(x)=c$$

$$g=y\sin x+y^2+k(x)=c$$

$$M=\frac{\partial g}{\partial x}=y\cos x+k'(x)=y\cos x$$

$$\Rightarrow k'(x)=0\ ,\ k(x)=\text{constant}$$

$$\therefore\ g(x,y)=y\sin x+y^2=c$$

- If $M(x,y)dx+N(x,y)dy=0$ but $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

$\Rightarrow$ multiply the ODE by $\mu(x,y)$

Assume "$\mu Mdx+\mu Ndy=0$" to be an "exact" ODE

So $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$

Two useful form for $\mu(x,y)$: $\mu(x)$ or $\mu(y)$

Ex: Given $\underbrace{(3xy+y^2)}_{M}+\underbrace{(x^2+xy)}_{N}\frac{dy}{dx}=0$, find $g(x,y)=?$

Sol:

Check: $\frac{\partial M}{\partial y}=3x+2y,\quad \frac{\partial N}{\partial x}=2x+y\ \Rightarrow\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, not exact

Multiply the D.E. by $\mu(x)$, integration factor

$$\mu(x)(3xy+y^2)+\mu(x)(x^2+xy)\frac{dy}{dx}=0$$

Let $\frac{\partial}{\partial y}[\mu(3xy+y^2)]=\frac{\partial}{\partial x}[\mu(x^2+xy)]$ $\langle\text{note}:\mu=\mu(x)\text{only}\rangle$

$$\Rightarrow\mu(3x+2y)=\mu'(x^2+xy)+\mu(2x+y)$$

$$\Rightarrow\mu(\cancel{x+y})=\mu'x(\cancel{x+y}),\qquad \mu=\mu'x=\frac{d\mu}{dx}x$$

$$\Rightarrow\int\frac{d\mu}{\mu}=\int\frac{dx}{x},\ \ \ln\mu=\ln x,\ \ \mu=x$$

So the D.E. becomes

$$\underbrace{(3xy+xy^2)}_{M}+\underbrace{(x^3+x^2y)}_{N}\frac{dy}{dx}=0$$

Check: $\frac{\partial M}{\partial y}=3x^2+2xy,\ \ \frac{\partial N}{\partial x}=3x^2+2xy\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},$ "exact" indeed !

$$\therefore g(x,y)=\int M(x,y)dx+f(y)=\text{const.}$$

$$\Rightarrow g(x,y)=\int\left[3x^2y+xy^2\right]dx+f(y)=c$$

$$\Rightarrow g(x,y)=x^3y+\frac{1}{2}x^2y^2+f(y)=c$$

thus $\frac{\partial g}{\partial y}=x^3+x^2y+\frac{\partial f}{\partial y}=N=x^3+x^2y\Rightarrow\frac{\partial f}{\partial y}=0,\quad f=\text{const.}$

$$\therefore g(x,y)=x^3y+\frac{1}{2}x^2y^2=\text{const.}$$

Summary:

For the D.E.: $M(x,y)+N(x,y)\frac{dy}{dx}=0$ (*)

If this D.E. is not exact, i.e. , $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

Find the integration factor μ to let (*) be exact

$$\Rightarrow \mu Mdx+\mu Ndy=0$$

1) Require: $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$

2) Step A: Try $\mu(x)$ only

Step B: Try $\mu(y)$ only

Step C: Guess $\mu(x,y)$

Example: $\underbrace{y}_{M}+\underbrace{\left(2x-ye^y\right)}_{N}\frac{dy}{dx}=0$

$$\Rightarrow\frac{\partial M}{\partial y}=1,\ \frac{\partial N}{\partial x}=2\ \Rightarrow\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\text{ , not exact}$$

1) let $\frac{\partial}{\partial y}(\mu M)=\frac{\partial}{\partial x}(\mu N)$, or $\underbrace{\frac{\partial}{\partial y}[\mu y]}_{\text{LHS}}=\underbrace{\frac{\partial}{\partial x}\left[\mu\left(2x-ye^y\right)\right]}_{\text{RHS}}$

2) Step A: Try $\mu=\mu(x)$ only, then

LHS= $\frac{\partial}{\partial y}[\mu y]=\mu(x)$

RHS= $\mu'\left(2x-ye^y\right)+2\mu$

LHS=RHS $\Rightarrow\mu'\left(2x-ye^y\right)+2\mu=\mu$

$$\Rightarrow \mu'\left(2x - ye^{y}\right) = -\mu$$

$$\Rightarrow -\frac{d\mu}{\mu} = \frac{dx}{2x - ye^{y}}$$

Cannot find the solution for $\mu(x)$!

Step :. Try $\mu = \mu(y)$, then

$$\text{LHS} = \mu' y + \mu = \text{RHS} = \mu(2)$$

$$\Rightarrow y\frac{d\mu}{dy} = \mu$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{dy}{y},\ \ln\mu = \ln y,\ \mu = \mu(y) = y$$

So the D.E. becomes

$$\underbrace{y^{2}}_{M} dx + \underbrace{\left(2xy - y^{2}e^{y}\right)}_{N} dy = 0$$

Let $g(x,y) = \int M\,dx + f(y) = \int y^{2}dx + f(y) = xy^{2} + f(y) = c$

$$\Rightarrow \frac{\partial g}{\partial y} = 2xy + \frac{\partial f}{\partial y} = N = 2xy - y^{2}e^{y}$$

$$\Rightarrow \frac{\partial f}{\partial y} = -y^{2}e^{y},\ f(y) = -\int y^{2}e^{y}dy$$ (Using the integration by parts)

$$\left[u = y^{2}, dv = e^{y}dy \Rightarrow v = e^{y}, du = 2y\,dy\right]$$

So $f(y) = -[uv]_{0}^{y} + \int_{0}^{y} v\,du$

$$= -y^{2}e^{y} + 2\int ye^{y}dy$$ [using the integration by parts]

$$\left[u = y, dv = e^{y}dy \Rightarrow v = e^{y}, du = dy\right]$$

$$= -y^{2}e^{y} + 2\left[ye^{y} - \int e^{y}dy\right]$$

$$= -y^{2}e^{y} + 2\left(ye^{y} - e^{y}\right) = e^{y}\left(-y^{2} + 2y - 2\right)$$

Thus its corresponding $g(x,y)$ is

$$g(x,y) = xy^{2} + e^{y}\left(-y^{2} + 2y - 2\right) = \text{const.}$$

- Review of Types of first-order ODE: $\frac{dy}{dx} = f(x,y)$

1) Separable: $f(x,y) = \frac{g(x)}{h(y)}$

2) Homogeneous: $f(x,y) = F\left(\frac{y}{x}\right)$

3) Linear: $f(x,y) = -a(x)y + b(x)$

4) Bernoulli's: $f(x,y) = -a(x)y + b(x)y^n$

5) Exact: $f(x,y) = -\frac{M(x,y)}{N(x,y)}$ such that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- Consider the applications of O.D.E. in the Population Model

Let $\rho =$ population of a species

$\frac{d\rho}{dt} =$ growth or death rate of a species

First consider "birth" only

birth rate $\propto$ population

$= \alpha P$, where α is the proportionality

$$\Rightarrow \frac{dP}{dt} = \alpha P$$

$$\Rightarrow \int \frac{dP}{P} = \int \alpha\, dt, \text{ or } \ln P = \alpha t + c,\ P = e^{\alpha t} \underbrace{e^{c}}_{c_1} = c_1 e^{\alpha t}$$

$\Rightarrow P = c_1 e^{\alpha t}$ where $P(0) = P_0 = c_1$

$\Rightarrow \boxed{P = P_0 e^{\alpha t}}$ Exponential growth of population!

Note that for radiation decay case, $\alpha < 0$ (negative trend)

Second, further consider "death due to competition"

death rate per individual $= -\beta P$

birth rate per individual $= \alpha$

$\boxed{\frac{dP}{P} = \alpha - \beta P}$ Logistic Curve

$$\frac{dP}{dt} = \alpha P - \beta P^2$$

$$\Rightarrow \int \frac{dP}{P(\alpha - \beta P)} = \int dt$$

$$\text{LHS} = \frac{1}{P(\alpha - \beta P)} = \frac{c_1}{P} + \frac{c_2}{\alpha - \beta P} = \frac{c_1\alpha + (c_2 - \beta c_1)P}{P(\alpha - \beta P)}$$

$$c_1\alpha = 1,\ c_2 = \beta c_1 \Rightarrow c_1 = \frac{1}{\alpha},\ c_2 = \beta c_1 = \frac{\beta}{\alpha}$$

$$\therefore \text{LHS} = \int \frac{1}{\alpha}\left[\frac{1}{P} + \frac{\beta}{\alpha - \beta P}\right] dP, \qquad \text{RHS} = t + \text{const.}$$

$$\text{LHS=RHS} \ \Rightarrow \ln|P| - \ln|\alpha - \beta P| = \alpha t + c$$

$$\Rightarrow \ln\left|\frac{P}{\alpha - \beta P}\right| = \alpha t + c$$

$$\Rightarrow \frac{P}{\alpha - \beta P} = e^{\alpha t} e^{c} = c^* e^{\alpha t}$$

$$\Rightarrow P = (\alpha - \beta P)c^* e^{\alpha t}$$

$$\Rightarrow P(t) = \frac{c^* \alpha\, e^{\alpha t}}{1 + c^* \beta\, e^{\alpha t}}$$

IC: $P(0) = P_0$ at time $t = 0$ $\Rightarrow P_0 = \dfrac{c^*\alpha}{1 + c^*\beta}$

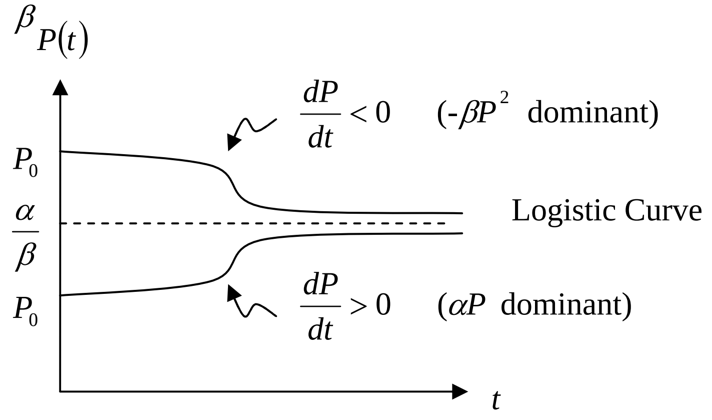
$$\Rightarrow c^*\alpha = (1 + c^*\beta)P_0$$

$$\Rightarrow c^* = \frac{P_0}{\alpha - \beta P_0}$$

Thus the population is

$$P(t) = \frac{\dfrac{P_0}{\alpha - \beta P_0}\alpha\, e^{\alpha t}}{1 + \dfrac{P_0}{\alpha - \beta P_0}\beta\, e^{\alpha t}} = \frac{\alpha P_0\, e^{\alpha t}}{(\alpha - \beta P_0) + \beta P_0\, e^{\alpha t}} = \frac{\alpha P_0\, e^{\alpha t}}{\alpha + \beta P_0\left(e^{\alpha t} - 1\right)}$$

and $\displaystyle\lim_{t\to\infty} P(t) = \frac{\alpha}{\beta}$



Problem for Predictor-Prey:

Ex: let *W*=number of wolves, *R*=number of rabbits

$$\begin{cases} \frac{1}{W}\frac{dW}{dt} = \alpha R - \beta & (1) \\ \frac{1}{R}\frac{dR}{dt} = a - bW & (2) \end{cases}$$

System of 1^{st} order equations

Find $W(R) = ?$

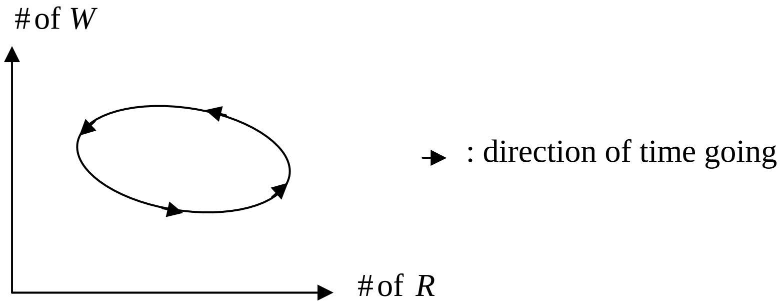
Sol: let $\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$ (Chain Rule)

$$\Rightarrow W(\alpha R - \beta) = \frac{dW}{dR}\left[R(a - bW)\right]$$

$$\Rightarrow \frac{dW}{dR} = \frac{W(\alpha R - \beta)}{R(a - bW)}$$

$$\Rightarrow \int \left[\frac{a - bW}{W}\right] dW = \int \left[\frac{\alpha R - \beta}{R}\right] dR$$

$$\Rightarrow a \ln W - bW = \alpha R - \beta \ln R + \text{const.}$$



● Nth order Linear Ordinary Differential Equations:

$$\frac{d^n y}{dt^n} + a_1(x)\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n(x)y = f(x)$$

The D.E. is linear in terms of *y* derivative

If $f(x) = 0 \Rightarrow$ this D.E. is said to be <u>homogeneous.</u>

Theorem: If $a_i(x)$ are continuous in $[a,b]$ and we have the above D.E. then

there exists a unique solution $y(x)$ in $[a,b]$ which satisfies D.E.

and the *n* initial conditions.

$$y(a) = c_0,\ y'(a) = c_1, \cdots, y^{(n-1)}(a) = c_{n-1}$$

<u>Def</u>: <u>Linear Independence:</u>

If we are given *n* functions, $f_1(x),\ \ f_2(x),\ \ f_3(x)$ then these n functions are said to be <u>linear independent</u> on some interval $[a,b]$, if on that interval

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \in [a,b]$$

Implies that $c_1 = c_2 = \cdots = c_n = 0$.

Otherwise the *n* functions are <u>linearly dependent</u> on $[a,b]$

The essential feature of linear equations is that:

Given any two solutions of the homogeneous equation, any linear combination of these two solutions is also a solution !

Proof: For the linear D.E.: $L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$

If $y_1(x)$ is a solution for $L(y) = 0$, and $y_2(x)$ is a solution

Then $y_3(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution too

$$\begin{aligned} \because L(c_1 y_1 + c_2 y_2) &= \left(c_1 y_1^{(n)} + c_2 y_2^{(n)}\right) + a_1(x)\left(c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}\right) + \cdots \\ &\quad + a_n(x)(c_1 y_1 + c_2 y_2) \\ &= c_1\left(y_1^{(n)} + a_1(x)y_1^{(n-1)} + \cdots + a_n(x)y_1\right) \\ &\quad + c_2\left(y_2^{(n)} + a_1(x)y_2^{(n-1)} + \cdots + a_n(x)y_2\right) \\ &= c_1 L(y_1) + c_2 L(y_2) = 0 + 0 = 0 \end{aligned}$$

Given nth order homogeneous ODE and *n* special solutions, any solution of the ODE is some linear combination of these *n* solutions, $y_1(x),\ \ y_2(x),\cdots,\ \ ,y_n(x)$.

Choose $y_1(x)$ solves the ODE and satisfies these *n* initial conditions,

$$y_1(a)=1,\quad y_1{}'(a)=y_1{}''(a)=\cdots=y_1{}^{(n-1)}(a)=0$$

$y_2(x)$ solves the ODE and satisfies these *n* initial conditions,

$$y_2(a)=0,\quad y_2{}'(a)=1,\ y_2{}''(a)=\cdots=y_2{}^{(n-1)}(a)=0$$

$\vdots$

$y_j(x)$ solves the ODE and satisfies these *n* initial conditions,

$$y_j(a)=0=y_j{}'(a)=\cdots=0,\quad y_j{}^{(j-1)}(a)=1,\quad y_j{}^{(j)}(a)=0=\cdots=y_j{}^{(n-1)}(a)$$

$\vdots$

i.e., ODE: $L(y)=y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\cdots+a_n(x)y=0$

ICs: $y(a)=c_1,\ y'(a)=c_2,\cdots,\ y^{(n-1)}(a)=c_n$

$\Rightarrow$ the general solution for this ODE is

$$y(x)=c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)$$

- Wronskian:

$$W[y_1,y_2,\cdots,y_n](x)=\det\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1{}'(x) & y_2{}'(x) & \cdots & y_n{}'(x) \\ \vdots & \vdots & & \vdots \\ y_1{}^{(n-1)}(x) & y_2{}^{(n-1)}(x) & \cdots & y_n{}^{(n-1)}(x) \end{vmatrix} \qquad (5.6)$$

If $y_1(x),\ \ y_2(x),\cdots,y_n(x)$ are n linearly independent solutions of the ODE:

$$y^{(n)}(x)+a_1(x)y^{(n-1)}(x)+\cdots+a_n(x)y=0$$

Then $\quad W'(x)+a_1(x)W(x)=0$

i.e., $\quad W(x)=W_0e^{-\int_0^x a(\tilde{x})d\tilde{x}} \qquad (5.7)\qquad$ Abel's formula

Note that

$W(x) \equiv 0$ if $W_0 = 0$ i.e., <u>n functions</u> $[(y_1(x), y_2(x), \cdots, y_n(x))]$ are linearly dependent (L.D.)

$W(x) \neq 0$ if $W_0 \neq 0$ i.e., n functions are linearly independent (L.I.)

Ex: $L(y) = y'' - \frac{1+x}{x} y' + \frac{1}{x} y = 0,\ x \in [1, \infty)$ (*)

Two solutions are $y_1(x) = 1 + x,\ y_2(x) = e^x$

Sol: Given $y_1(x)$ and $y_2(x)$, check whether they satisfy ODE (*) or not ?

$$\Rightarrow y_1{}'(x) = 1,\ y''(x) = 0, L(y_1) = 0 - \frac{1+x}{x}(1) + \frac{1}{x}(1+x) = 0$$

$$y_2{}'(x) = e^x,\ y''(x) = e^x, L(y_2) = e^x - \frac{1+x}{x} e^x + \frac{1}{x}\left(e^x\right) = 0$$

Check $y_1(x)$ and $y_2(x)$ are L.I. or not ?

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1+x & e^x \\ 1 & e^x \end{vmatrix} = xe^x \neq 0 \text{ on } [1, \infty)$$

$\therefore$ any solution for ODE (*) can be written as

$$y(x) = c_1(1+x) + c_2 e^x$$

- Review of "Linear Independence" (L.I.) of the ODE:

—Essential Feature (E.F.) of Linear Equations:

L.I.: $c_1 y_1(x) + c_2 y_2(x) = 0 \Rightarrow c_1 = c_2 = 0$

E.F.: 1) if y_1, y_2 are solutions of ODE $\Rightarrow c_1 y_1 + c_2 y_2$ is also a solution of ODE

2) if $\begin{cases} y_1 \text{ and } y_2 \text{ are solutions of the 2nd oder ODE} \\ y_1 \text{ and } y_2 \text{ are linearly independent} \end{cases}$

then $y_3 = c_1 y_1 + c_2 y_2$ is the general solution for the ODE

2nd-Order Constant-Coefficient Equations:

For homogeneous equation

$$\begin{cases} \text{D.E.: } y''(x) + Ay' + By = 0 \\ \text{I.C.: } y(a) = c_1,\ y'(a) = c_2 \end{cases} \qquad (5.8)$$

Let $y(x) = e^{\alpha x} \Rightarrow y'(x) = \alpha e^{\alpha x},\ y''(x) = \alpha^2 e^{\alpha x}$

Substituting into D.E. i.e., Eq.(5.8):

$(\alpha^2 + A\alpha + B)e^{\alpha x} = 0$, consider nontrivial solution (i.e., $e^{\alpha x} \neq 0$)

$\Rightarrow \alpha^2 + A\alpha + B = 0$, the characteristic equation

$$\text{or}\quad \alpha_{\pm} = \frac{-A}{2} \pm \frac{\sqrt{A^2 - 4B}}{2} \qquad (5.9)$$

Two solutions of $y(x)$:

$$y_1(x) = e^{\alpha_+ x},\ y_2(x) = e^{\alpha_- x} \qquad (5.10)$$

General solution for D.E.:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad (5.11)$$

Check whether y_1 and y_2 are linearly independent (L.I.) ?

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\alpha_1 x} & e^{\alpha_2 x} \\ \alpha_1 e^{\alpha_1 x} & \alpha_2 e^{\alpha_2 x} \end{vmatrix} = (\alpha_2 - \alpha_1) e^{(\alpha_1 + \alpha_2)x}$$

So $W \neq 0$ if $\alpha_1 \neq \alpha_2$, then y_1 and y_2 are L.I.

Since $\alpha_+ = \frac{-A}{2} + \frac{\sqrt{A^2 - 4B}}{2}$, $\alpha_- = \frac{-A}{2} - \frac{\sqrt{A^2 - 4B}}{2}$

For $\alpha_+ = \alpha_- \Rightarrow A^2 - 4B = 0$, then y_1 and y_2 are L.D.

Then we need another way to find a L.I. y_2 $\left(\text{in this case, } y_1 = e^{-\frac{A}{2}x}\right)$

- D.E. : $y'' + Ay' + B = 0$

  3 cases:

  1) $A^2 - 4B > 0$: α_+, α_- are distinct and real

  2) $A^2 - 4B < 0$: α_+, α_- are distinct and complex

  3) $A^2 - 4B = 0$: $\alpha_+ = \alpha_-$, repeat roots

Case Ⅰ) $\underline{A^2 - 4B > 0:}$

General solution $y(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}$

Ex: $y'' + 5y' + 4y = 0$

Sol: let $y = e^{\alpha x} \Rightarrow \alpha^2 + 5\alpha + 4 = 0 \Rightarrow (\alpha+1)(\alpha+4) = 0 \Rightarrow \alpha = -4 \text{ or } -1$

$\therefore\ y_1 = e^{-4x},\ y_2 = e^{-x}$

general solution is $y = c_1 e^{-4x} + c_2 e^{-x}$

Case Ⅱ) $\underline{A^2 - 4B < 0:}$

$\Rightarrow \sqrt{A^2 - 4B}$ is imaginary

$$\Rightarrow \alpha_\pm = \frac{-A}{2} \pm i\frac{\sqrt{4B - A^2}}{2}$$

$$\left.\begin{aligned} e^{\alpha_+ x} &= e^{-\frac{A}{2}x} e^{i\frac{\sqrt{4B-A^2}}{2}x} \\ e^{\alpha_- x} &= e^{-\frac{A}{2}x} e^{-i\frac{\sqrt{4B-A^2}}{2}x} \end{aligned}\right\} \quad y(x) = \underline{c_1} e^{\alpha_+ x} + \underline{c_2} e^{\alpha_- x}$$

complex constant

Define: $D = \dfrac{\sqrt{4B - A^2}}{2}$

and $e^{iDx} = \cos(Dx) + i\sin(Dx)$

$e^{-iDx} = \cos(Dx) - i\sin(Dx)$

$$\Rightarrow \cos(Dx) = \frac{1}{2}\left(e^{iDx} + e^{-iDx}\right)$$

$$\sin(Dx) = \frac{1}{2i}\left(e^{iDx} - e^{-iDx}\right)$$

$$\Rightarrow y_1(x) = e^{-\frac{A}{2}x}\cos(Dx), \quad y_2(x) = e^{-\frac{A}{2}x}\sin(Dx) \qquad (5.12)$$

general solution is

$$y(x) = \underline{c_1} y_1(x) + \underline{c_2} y_2(x)$$

real constant

Case III) $A^2 - 4B = 0$:

$$\alpha_+ = \alpha_- = \frac{-A}{2}$$

$$y_1(x) = e^{\frac{-A}{2}x} \quad \text{the only solution we got}$$

$$\Rightarrow \quad y_2(x) = xe^{\frac{-A}{2}x} \quad \text{the second solution obtained by the method of reduction of order} \qquad (5.13)$$

The Method of Reduction of Order:

ODE: $y''(x) + a(x)y'(x) + b(x) = 0$ (5.14)

Given $y_1(x)$ is a solution to the above ODE

Assume $y_2(x) = y_1(x)v(x)$ is also a solution to this ODE

$$\Rightarrow \quad y_2'(x) = y_1'v + y_1v'$$

$$y_2''(x) = y_1''v + 2y_1'v' + y_1v''$$

Substitution into the ODE, i.e. Eq. (5.14):

$$\left(y_1''v + 2y_1'v' + y_1v''\right) + a\left(y_1'v + y_1v'\right) + by_1v = 0$$

$$\Rightarrow v\underbrace{\left(y_1'' + ay_1' + by_1\right)}_{0} + \underbrace{v'\left(2y_1' + ay_1\right) + y_1v''}_{(*)} = 0$$

$\because y_1$ is the solution (*)

So the D.E. (*) is first order D.E. in $u(x) = v'(x)$ $\quad (u'(x) = v''(x))$

(*): $\quad y_1u' + \left(2y_1' + ay_1\right)u = 0$

$$y_1\frac{du}{dx} = -\left(2y_1' + ay_1\right)u$$

A separable D.E. for *u(x)*:

$$\Rightarrow \int \frac{du}{u} = -\int\left[\frac{2y' + ay_1}{y_1}\right]dx$$

$$\Rightarrow \ln|u| = -2\ln|y_1| - \int_0^x a(\tilde{x})d\tilde{x}$$

$$\Rightarrow \text{ or } \quad u(x) = \left(y_1(x)\right)^{-2}\exp\left[-\int_0^x a(\tilde{x})d\tilde{x}\right]$$

$$u(x) = \frac{1}{\left(y_1(x)\right)^2}\exp\left[-\int_0^x a(\tilde{x})d\tilde{x}\right]$$

and

$$v'(x) = u(x)$$

$$\Rightarrow \qquad v(x) = \int_0^x u(x^*)dx^*$$

$$= \int_0^x \frac{1}{y_1^2(x^*)} \exp\left[-\int_0^{x^*} a(\tilde{x})d\tilde{x}\right]dx^* \qquad (5.15)$$

$$y_2(x) = y_1(x)v(x)$$

Back to the Repeated Root Case of Case Ⅲ:

ODE: $y'' + Ay' + By = 0$ (5.8)

For repeated root $\Rightarrow A^2 = 4B$

$$y_1(x) = e^{\frac{-A}{2}x}$$

then let us find $y_2(x)$ by the method of "reduction of order"

assume $y_2(x) = y_1(x)v(x) = e^{\frac{-A}{2}x}v(x)$

$$\Rightarrow y_2' = \frac{-A}{2}e^{\frac{-A}{2}x}v + e^{\frac{-A}{2}x}v'$$

$$\Rightarrow y_2'' = \frac{-A}{2}\left[\frac{-A}{2}e^{\frac{-A}{2}x}v + e^{\frac{-A}{2}x}v'\right] + \left[\frac{-A}{2}e^{\frac{-A}{2}x}v' + e^{\frac{-A}{2}x}v''\right]$$

Substitution into the ODE, Eq. (5.8):

$$e^{\frac{-A}{2}x}\left\{\left(\frac{A^2}{4}v - \frac{A}{2}v' - \frac{A}{2}v' + v''\right) + A\left(\frac{-A}{2}v + v'\right) + Bv\right\} = 0$$

$$\Rightarrow v''(x) = 0$$

$$\Rightarrow v(x) = ax + b$$

$$\therefore y_2(x) = y_1(x)v(x) = (ax+b)e^{\frac{-A}{2}x}$$

Without lost of generality, let us set $a = 1,\ b = 0$

$$\therefore y_2(x) = xe^{\frac{-A}{2}x}$$

Example: $a_0(x)y'' + (Ax + B)y' - Ay = 0$ (A)

Sol: for non-constant coefficient $\Rightarrow$ guess simple solution first

$1, x, x^2, \cdots, x^n, e^{\alpha x}$

Guess $y_1(x) = \alpha\, x + \beta$ (B)

Substitution (B) into D.E. (A) to find α, β

$\Rightarrow y_2(x) = y_1(x)v(x)$ then solve for $v(x)$

$\Rightarrow$ thus $y(x) = c_1 y_1 + c_2 y_2$ for the general solution to (A)

(A): $(Ax+B)\alpha - A(\alpha\, x + \beta) = 0$

$$B\alpha = A\beta \Rightarrow \frac{A}{B} = \frac{\alpha}{\beta}$$

$\therefore$ take $y_1(x) = Ax + B$

set $y_2(x) = v(x)y_1 = v(x)(Ax+B)$

$\Rightarrow y_2' = v'(Ax+B) + vA$ (C)

$y_2''' = v''(Ax+B) + 2vA$ (D)

Substitution (C), (D) into (A) :

$$a_0(x)(v''(Ax+B) + 2v'A) + (Ax+B)(v'(Ax+B) + vA) - Av(Ax+B) = 0$$

$$\Rightarrow a_0(x)(Ax+B)v'' + \left(2Aa_0(x) + (Ax+B)^2\right) = 0 \qquad \text{(E)}$$

let $v'(x) = u(x)$

So Eq.(E) becomes $a_0(x)(Ax+B)u' + \left[2Aa_0(x) + (Ax+B)^2\right]u = 0$

$$\Rightarrow a_0(x)(Ax+B)\frac{du}{dx} = -\left[2Aa_0(x) + (Ax+B)^2\right]u$$

Separable: $$\int \frac{du}{u} = \int -\left[\frac{2Aa_0(x) + (Ax+B)^2}{a_0(x)(Ax+B)}\right]dx$$

$$\Rightarrow \ln u = -\int \frac{2A}{Ax+B}dx - \int \frac{Ax+B}{a_0(x)}dx$$

$$\ln u = -2\ln(Ax+B) - \int_0^x \left[\frac{Ax+B}{a_0(x)}\right]dx$$

$$\Rightarrow u(x) = (Ax+B)^{-2}\exp\left[-\int_0^x \frac{A\tilde{x}+B}{a_0(\tilde{x})}d\tilde{x}\right]$$

and $v'(x) = u$ $\left(\because v = \int u(x)dx\right)$

$$\Rightarrow v(x) = \int_0^x (Ax^*+B)^{-2}\exp\left[-\int_0^{x^*} \frac{A\tilde{x}+B}{a_0(\tilde{x})}d\tilde{x}\right]dx^*$$

Example: $(t^2+1)\dfrac{d^2y}{dt^2} - 2t\dfrac{dy}{dt} + 2y = 0$ (*), find $y(t) = ?$

Sol: $\begin{cases} a_0(t)\dfrac{d^2y}{dt^2}+(At+B)\dfrac{dy}{dt}-Ay=0 \\ \text{let} \quad y_1(t)=At+B \end{cases}$

$\therefore y_1(t)=t$ is a solution to the D.E. (*)

let $y_2(t)=y_1(t)v(t)=tv(t)$

$$\Rightarrow y_2'(t)=v't+v, \quad y_2''(t)=v''t+2v'$$

Substitution y_2 into the D.E. (*):

$$(t^2+1)(v''t+2v')-2t(v't+v)+2tv=0$$
$$\Rightarrow (t^2+1)tv''+\left[2(1+t^2)-2t^2\right]v'=0$$
$$\Rightarrow v''+\frac{2}{t(1+t^2)}v'=0$$

Let $u(t)=v'(t)$

$$\Rightarrow u'+\frac{2}{t(1+t^2)}u=0$$
$$\Rightarrow \int\frac{du}{u}=\int\frac{-2}{t(1+t^2)}dt$$
$$\Rightarrow \ln|u|=\int\left[\frac{A}{t}+\frac{Bt+C}{1+t^2}\right]dt$$
$$\Rightarrow \ln|u|=\int\left[\frac{-2}{t}+\frac{2t}{1+t^2}\right]dt \quad (A=-2, B=2, C=0)$$
$$\Rightarrow \ln|u|=-2\ln|t|+\ln\left|1+t^2\right|=\ln t^{-2}+\ln(1+t^2)$$
$$\Rightarrow u(t)=\frac{1+t^2}{t^2}$$
$$\Rightarrow v(t)=\int u(t)dt=\int\left(\frac{1+t^2}{t^2}\right)dt=\frac{-1}{t}+t$$

$$\therefore y_2(t)=y_1(t)v(t)=t\left(\frac{-1}{t}+t\right)=t^2-1$$

Thus the general solution is

$$y(t)=c_1y_1(t)+c_2y_2(t)=c_1t+c_2(t^2-1)$$

● For 2nd-order Non-homogeneous Differential Equations:

$$y''(x)+a(x)y'(x)+b(x)y(x)=f(x) \qquad (5.16)$$

Known $y_1(x)$ is a solution of the homogeneous equation

Then by the method of "reduction of order"

$$\Rightarrow \quad y_2(x) = y_1(x)v(x)$$

$$\Rightarrow \quad y_2'(x) = y_1'v + y_1v'$$

$$\Rightarrow \quad y_2''(x) = y_1''v + 2y_1'v' + y_1v''$$

Substitution into the D.E., i.e., Eq. (5.16)

$$\left(y_1''v + 2y_1'v' + y_1v''\right) + a(x)\left(y_1'v + y_1v'\right) + b(x)y_1v = f(x)$$

$$\Rightarrow v\left[y_1'' + a(x)y' + b(x)y_1\right] + v'\left[2y_1' + a(x)y_1\right] + y_1v'' = f(x)$$

$\because y_1$ is the solution, so the first term in the bracket vanishes

$$\Rightarrow y_1v'' + \left[2y_1' + ay_1\right]v' = f(x) \qquad (5.17)$$

A linear 1st-order D.E. for $u(x) = v'(x)$

Ex: $xy'' - xy' + y = x^2$, find $y(x) = ?$

Sol: A obvious solution to the homogeneous D.E. is

$$y_1(x) = x$$

So let $y_2(x) = y_1(x)v(x) = xv(x)$

$$\Rightarrow y_2' = v'x + v, \quad y_2'' = v''x + 2v'$$

Substitution y_2 into the D.E.:

$$x(v''x + 2v') - x(v'x + v) + vx = x^2$$

$$\Rightarrow x^2v'' + \left(2x - x^2\right)v' = x^2$$

$$\Rightarrow v'' + \left(\frac{2}{x} - 1\right)v' = 1 \text{ or } \underbrace{u' + \left(\frac{2}{x} - 1\right)u = 1,}_{(*)} \quad u = v'$$

Try the integration-factor method to solve $u(x)$ is (*):

$$\mu(x) = \exp\left[\int\left(\frac{2}{x} - 1\right)dx\right] = \exp[2\ln x - x] = x^2e^{-x}$$

$$(*) \text{ becomes}: \frac{d}{dx}\left(x^2e^{-x}u\right) = x^2e^{-x}$$

$$\Rightarrow x^2e^{-x}u = \int x^2e^{-x}dx = e^{-x}\left(-x^2 - 2x - 2\right) + c_1$$

$$\Rightarrow u(x) = v'(x) = \frac{c_1}{x^2e^{-x}} + \frac{e^{-x}}{x^2e^{-x}}\left(-x^2 - 2x - 2\right)$$

$$= c_1 \frac{e^x}{x^2} - \left(1 + \frac{2}{x} + \frac{2}{x^2}\right)$$

$$\Rightarrow v(x) = \int u(x)dx = c_2 + c_1 \int \frac{e^x}{x^2} dx - \int \left(1 + \frac{2}{x} + \frac{2}{x^2}\right) dx$$

$$= c_2 + c_1 \int \frac{e^x}{x^2} dx - x - 2\ln|x| + \frac{2}{x}$$

$$\therefore \begin{cases} y_2(x) = y_1(x)v(x) = xv(x) \\ \qquad = c_2 x + c_1 x \int \frac{e^x}{x^2} dx - x^2 - 2x\ln|x| + 2 \\ y_1(x) = x \end{cases}$$

general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$

- Homogeneous and Particular Solutions:

For the Nonhomogeneous Differential Equation:

$$L(y) = y''(x) + a(x)y'(x) + b(x)y(x) = f(x)$$

1) First look at the homogeneous equation
   a) Find just one solution to the homogeneous eq., i.e., find $y_1(x)$ s.t. $L(y_1) = 0$
   b) Find another solution $y_2(x)$, i.e., find $y_2(x)$ s.t. $L(y_2) = 0$
      then $y_H(x) = c_1 y_1(x) + c_2 y_2(x)$, if y_1, y_2 are L.I. (Linear Independent)

2) Find any solution satisfiying non-homogeneous eq., i.e., $y_p(x)$, s.t. $L(y_p) = f(x)$

3) The general solution is $y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$

Check: $\because L(y_1) = 0, \quad L(y_2) = 0$

$$\Rightarrow L(y) = L(y_p + c_1 y_1 + c_2 y_2)$$

$$= \underbrace{L(y_p)}_{f(x)} + c_1 \underbrace{L(y_1)}_{0} + c_2 \underbrace{L(y_2)}_{0} = f(x)$$

Thus $y(x)$ is the solution for $L(y) = f(x)$

$$\therefore y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$

- Assume $y_1(x)$ satisfy D.E.(1): $y''(x) + a(x)y'(x) + b(x)y(x) = f_1(x)$
  Assume $y_2(x)$ satisfy D.E.(1): $y''(x) + a(x)y'(x) + b(x)y(x) = f_2(x)$

$\Rightarrow y_1(x) + y_2(x)$ satisfy: $y''(x) + a(x)y'(x) + b(x)y(x) = \underbrace{f_1(x)}_{\text{Forcing (1)}} + \underbrace{f_2(x)}_{\text{Forcing (2)}}$

<u>Method of Undetermined Coefficient</u>:

$$y''(x) + Ay'(x) + By(x) = f(x)$$

where

$$f(x) = \begin{cases} e^{\alpha x} \\ P_n(x)e^{\alpha x} \\ \sin\beta x,\ \cos\beta x \\ P_n(x)\sin\beta x, P_n(x)\cos\beta x \\ P_n(x)e^{\alpha x}(\sin\beta x, \cos\beta x) \end{cases}$$

We can guess the particular solution by the method of "undermined coefficient".

e.g., $f(x) = e^{\alpha x}\sin\beta x$

$\Rightarrow$ guess a particular solution as

$y(x) = e^{\alpha x}(A\sin\beta x + B\cos\beta x)$, then find out A and B

Ex: $\begin{cases} \text{D.E.: } y'' + y = 2\sin 2x \\ \text{I.C.: } y(0) = 1,\ y'(0) = -1 \end{cases} \Rightarrow$ find $y(x) = ?$

Sol: characteristic equation for $y'' + y = 0$

Let $y = e^{\alpha x} \Rightarrow \alpha^2 + 1 = 0 \quad \Rightarrow \alpha = \pm i$

Thus two homogeneous solutions: $y_1 = \sin x,\quad y_2 = \cos x$

Assume the general solution as $y = c_1 \sin x + c_2 \cos x + y_p$

Guess $y_p(x)$: Try $y_p = D_1 \sin 2x + D_2 \cos 2x$

$$\Rightarrow y_p' = 2D_1 \cos 2x - 2D_2 \sin 2x$$

$$\Rightarrow y_p'' = -4D_1 \sin 2x - 4D_2 \cos 2x$$

Substitution into D.E.: $y_p'' + y_p = 2\sin 2x$

$$\Rightarrow (-4D_1 \sin 2x - 4D_2 \cos 2x) + (D_1 \sin 2x + D_2 \cos 2x) = 2\sin 2x$$

$\cos 2x: -4D_2 + D_2 = 0 \quad \Rightarrow D_2 = 0$

$\sin 2x: -4D_1 + D_1 = 2 \quad \Rightarrow D_1 = -\dfrac{2}{3}$

$$\therefore y_p = -\frac{2}{3}\sin 2x$$

thus general solution is $y = c_1 \sin x + c_2 \cos x - \frac{2}{3}\sin 2x$

and $y' = c_1 \cos x - c_2 \sin x - \frac{4}{3}\cos 2x$

check with ICs : $y(0) = 1, \quad y'(0) = -1$

$\Rightarrow 1 = c_1(0) + c_2(1) - \frac{2}{3}(0) \quad \Rightarrow c_2 = 1$

$-1 = c_1(0) - c_2(0) - \frac{4}{3}(1) \quad \Rightarrow -1 = c_1 - \frac{4}{3}, \quad c_1 = \frac{1}{3}$

$\Rightarrow \quad y = \frac{1}{3}\sin x + \cos x - \frac{2}{3}\sin 2x$

- All linear Constant-Coefficient 2^{nd}-order Differential Equation can be solve by solving their characteristic equations

Example: $y'' + 3y' + 2y = x^2 e^{2x} \Rightarrow$ find $y(x) = ?$

Sol: Characteristic Eq: $\alpha^2 + 3\alpha + 2 = 0$

$$\Rightarrow (\alpha + 2)(\alpha + 1) = 0$$
$$\Rightarrow \alpha = -2, \ \alpha = -1$$

homogeneous solution: $y_1 = e^{-2x}, \quad y_2 = e^{-x}$

particular solution: assume $y_p = \left(c_1 x^2 + c_2 x + c_3\right)e^{2x}$

$$\Rightarrow y_p' = (2c_1 x + c_2)e^{2x} + 2\left(c_1 x^2 + c_2 x + c_3\right)e^{2x}$$
$$\Rightarrow y_p'' = 2c_1 e^{2x} + 2(2c_1 x + c_2)\cdot 2e^{2x} + \left(c_1 x^2 + c_2 x + c_3\right)\cdot 4e^{2x}$$

Rewrite:

$$2\cdot y_p = \left(c_1 x^2 + c_2 x + c_3\right)e^{2x}$$
$$3\cdot y_p' = \left[2c_1 x^2 + (2c_1 + 2c_2)x + (2c_2 + 2c_3)\right]e^{2x}$$
$$+) \quad 1\cdot y_p'' = \left[4c_1 x^2 + (8c_1 + 4c_2)x + (4c_3 + 4c_2 + 2c_1)\right]e^{2x}$$

$$\Rightarrow \quad x^2 e^{2x} : 4c_1 + 6c_1 + 2c_1 = 1 \quad \Rightarrow 12c_1 = 1, \ c_1 = \frac{1}{12}$$

$$xe^{2x} : (8c_1 + 4c_2) + 3(2c_1 + 2c_2) + 2c_2 = 0 \Rightarrow 14c_1 + 12c_2 = 0$$
$$\Rightarrow c_2 = -\frac{14}{12}c_1$$

$$e^{2x}:(4c_3+4c_2+2c_1)+3(2c_2+2c_3)+2c_3=0$$
$$\Rightarrow 12c_3+10c_2+2c_1=0$$
$$\Rightarrow c_3=-\frac{1}{12}(10c_2+2c_1)$$

thus $c_1=\frac{1}{12},\ c_2=-\frac{14}{12}\cdot\frac{1}{12}=-\frac{7}{72}$

$$c_3=-\frac{1}{12}\left[10\left(-\frac{7}{72}\right)+\frac{2}{12}\right]=-\frac{1}{12}\left[\frac{-70+12}{72}\right]=\left(-\frac{1}{12}\right)\left(-\frac{60}{72}\right)=\frac{5}{72}$$

$$\therefore y_p=\left(\frac{1}{12}x^2-\frac{7}{72}x+\frac{5}{72}\right)e^{2x}$$

and the general solution is

$$y(x)=c_1e^{-2x}+c_2e^{-x}+y_p$$

Example: $y''+3y'+2y=xe^{-x}\Rightarrow$ find $y(x)=?$

Sol: Characteristic Eq: $\alpha^2+3\alpha+2=0\Rightarrow(\alpha+2)(\alpha+1)=0\Rightarrow\alpha=-2,\ -1$

So homogeneous solution: $y_1=e^{-x},\quad y_2=e^{-2x}$

When the forcing term (e^{-x}) is some form of homogeneous solution, then we can guess the particular solution as

$$2\cdot y_p(x)=x\cdot(Ax+B)e^{-x}=(Ax^2+Bx)e^{-x}$$
$$3\cdot y_p'(x)=(2Ax+B)e^{-x}-(Ax^2+Bx)e^{-x}=\left[-Ax^2+(2A-B)x+B\right]e^{-x}$$
$$1\cdot y_p''(x)=2Ae^{-x}+2(2Ax+B)(-e^{-x})+(Ax^2+Bx)e^{-x}$$
$$+)\qquad =\left[Ax^2+(-4A+B)x+(2A-2B)\right]e^{-x}$$

$$\Rightarrow\quad x^2e^{-x}:A-3A+2A=0\quad\Rightarrow 0=0$$
$$xe^{-x}:(-4A+B)+3(2A-B)+2B=1\Rightarrow 2A=1\Rightarrow A=\frac{1}{2}$$
$$e^{-x}:(2A-2B)+3B=0\Rightarrow 2A+B=0\Rightarrow B=-2A=-2\left(\frac{1}{2}\right)=-1$$

$\Rightarrow A=\frac{1}{2},\quad B=-1$ and $y_p(x)=x\left(\frac{1}{2}x-1\right)e^{-x}=\left(\frac{x^2}{2}-x\right)e^{-x}$

Thus, the general solution is $y(x)=c_1e^{-x}+c_2e^{-2x}+\left(\frac{x^2}{2}-x\right)e^{-x}$

Note that if $f(x) = P_n(x)e^{\alpha x}$ (P$_n$ is a polynominal) and $y_1(x)$ or $y_2(x) = e^{\alpha x}$

1) then guess $y_p(x) = xQ_n(x)e^{\alpha x}$

2) if $f(x) = P_n(x)e^{\alpha x}$ and $y_1(x) = y_2(x) = e^{\alpha x}$ (repeated root)

then guess $y_p(x) = x^2 Q_n(x)e^{\alpha x}$

Example: $y'' + \omega^2 y = \sin(\omega x) \Rightarrow y(x) = ?$

Sol : homogeneous solution: $y_1(x) = \sin(\omega x),\ y_2(x) = \cos(\omega x)$

Using the method of undetermined coefficient$\Rightarrow$

let $y_p(x) = x(A\cos\omega x + B\sin\omega x)$

$$\Rightarrow y_p'(x) = (A\cos\omega x + B\sin\omega x) + x(-A\omega\sin\omega x + B\omega\cos\omega x)$$

$$\Rightarrow y_p''(x) = 2(-A\omega\sin\omega x + B\omega\cos\omega x) + x(-A\omega^2\cos\omega x - B\omega^2\sin\omega x)$$

$$= (-2A\omega - B\omega^2 x)\sin\omega x + (2B\omega - A\omega^2 x)\cos\omega x$$

$$\Rightarrow y_p'' + \omega^2 y_p = (-2A\omega - B\omega^2 x)\sin\omega x + (2B\omega - A\omega^2 x)\cos\omega x$$
$$+ (A\omega^2 x\cos\omega x + B\omega^2 x\sin\omega x)$$

$$\sin\omega x : -2A\omega - B\omega^2 x + B\omega^2 x = 1 \qquad \Rightarrow A = \frac{-1}{2\omega}$$

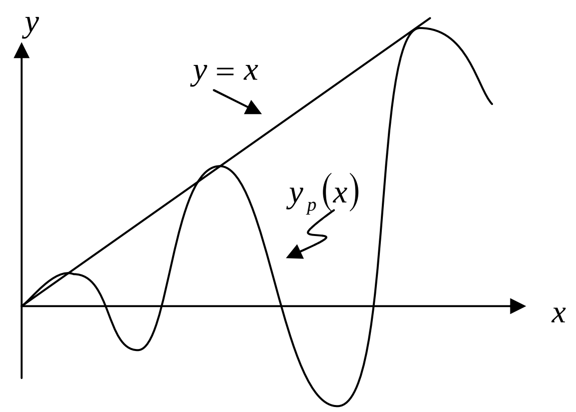
$$\cos\omega x : 2B\omega - A\omega^2 x + A\omega^2 x = 0 \qquad \Rightarrow B = 0$$

$$\therefore y_p = -\frac{1}{2\omega} x\cos\omega x$$

and the general solution is

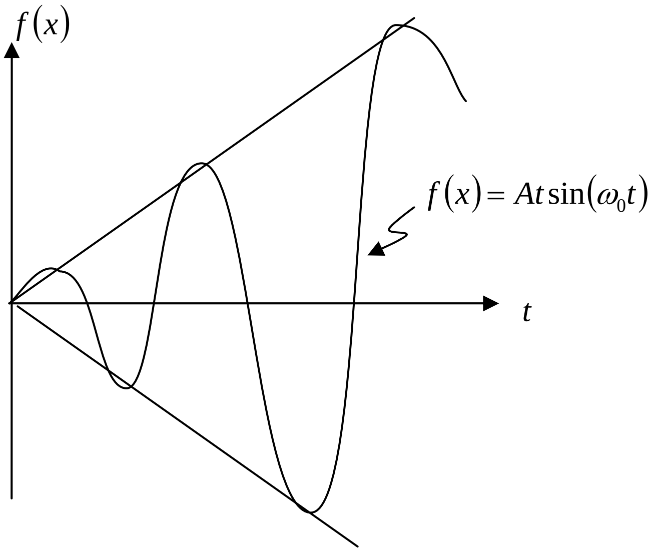
$$y = c_1\sin\omega x + c_2\cos\omega x - \frac{1}{2\varpi} x\cos\omega x$$

Graph of $y_p(x)$:



Resonance
Amplitude $\propto x$

Graph of $f(x) = At\sin(\omega_0 t)$:



- Method of Parameters (Constants)

ODE: $y''(x) + a(x)y'(x) + b(x)y(x) = f(x)$

Known 2 homogeneous solutions: $y_1(x), y_2(x)$

Let parameter solution be $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$

$$\Rightarrow y_p'(x) = (A'y_1 + Ay_1') + (B'y_2 + By_2')$$
$$= (A'y_1 + B'y_2) + (Ay_1' + By_2')$$

if we set $A'y_1 + B'y_2 = 0$

then $y_p' = Ay_1' + By_2'$

$$y_p'' = A'y_1' + Ay_1'' + B'y_2' + By_2''$$

Substitution into the ODE:

$$(A'y_1' + Ay_1'' + B'y_2' + By_2'') + a(x)(Ay_1' + By_2') + b(x)(Ay_1 + By_2) = f(x)$$

$$\Rightarrow A(y_1'' + ay_1' + by_1) + B(y_2'' + ay_2' + by_2) + A'y_1' + B'y_2' = f(x)$$

$\because L(y_1) = 0$ (0) $\quad \because L(y_2) = 0$ (0)

2 conditions must be satisfied:

$$\begin{cases} A'y_1 + B'y_2 = 0 \\ A'y_1' + B'y_2' = f(x) \end{cases}, \text{or in matrix form} \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

We can use the Cramer Rule to solve A' and B'

$$\Rightarrow A' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad B' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$= \frac{-y_2 f(x)}{W(y_1, y_2)} \qquad = \frac{y_1 f(x)}{W(y_1, y_2)} \quad \text{, where } W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$\Rightarrow A(x) = -\int_c^x \frac{y_2 f(\bar{x})}{W(y_1, y_2)} d\bar{x}, \qquad B(x) = \int_c^x \frac{y_1 f(\bar{x})}{W(y_1, y_2)} d\bar{x} \quad , c \text{ to be determined}$$

$$\therefore y_p(x) = A(x) y_1(x) + B(x) y_2(x)$$

i.e., $$y_p(x) = -y_1(x) \int_c^x \frac{y_2(\bar{x}) f(\bar{x})}{W(y_1, y_2)} d\bar{x} + y_2(x) \int_c^x \frac{y_1(\bar{x}) f(\bar{x})}{W(y_1, y_2)} d\bar{x} \qquad (5.18)$$

c can be determined by ICs

Ex: $xy'' - (1+x)y' + y = x^2 e^{2x} \Rightarrow$ find $f(x) = ?$

Sol: Guess two homogeneous solutions: $y_1(x) = 1 + x,\ y_2(x) = e^x$

then try particular solution as $y_p(x) = A(x)(1+x) + B(x) e^x$

$$\Rightarrow y_p'(x) = \underbrace{A'(1+x) + B' e^x}_{\text{assume} \equiv 0} + A + B e^x$$

$$\Rightarrow y_p''(x) = A' + B' e^x + B e^x$$

Substitution into the ODE:

$$\Rightarrow x(A' + B'e^x + Be^x) - (1+x)(A + Be^x) + A(1+x) + Be^x = x^2 e^{2x}$$

$$\Rightarrow \begin{cases} A'(1+x) + B'e^x = 0 & \text{(condition from the assumption)} \\ A' + B'e^x = xe^{2x} & \text{(condition from the substitution into the ODE)} \end{cases}$$

$$\Rightarrow \begin{pmatrix} 1+x & e^x \\ 1 & e^x \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 0 \\ xe^{2x} \end{pmatrix}$$

$$\Rightarrow \quad A' = \frac{\begin{vmatrix} 0 & e^x \\ xe^{2x} & e^x \end{vmatrix}}{\begin{vmatrix} 1+x & e^x \\ 1 & e^x \end{vmatrix}}, \quad B' = \frac{\begin{vmatrix} 1+x & 0 \\ 1 & xe^{2x} \end{vmatrix}}{\begin{vmatrix} 1+x & e^x \\ 1 & e^x \end{vmatrix}}$$

$$= \frac{-xe^{3x}}{xe^x} = -e^{2x} \qquad = \frac{x(1+x)e^{2x}}{xe^x} = (1+x)e^x$$

$$\Rightarrow\; A = -\int e^{2x}dx = -\frac{1}{2}e^{2x}$$

$$B = \int (1+x)e^{x}dx = xe^{x} \quad \left[\because \frac{d}{dx}\left(xe^{x}\right) = e^{x} + xe^{x} = (1+x)e^{x} \right]$$

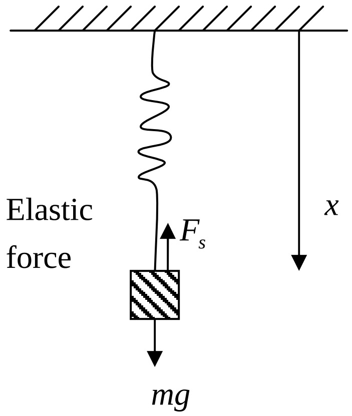
Thus the particular solution is

$$y_p = -\frac{1}{2}e^{2x}(1+x) + x\,e^{x}\cdot e^{x} = e^{2x}\left(-\frac{1+x}{2}+x\right) = \frac{1}{2}e^{2x}(x-1)$$

And the general solution is

$$y(x) = c_1(1+x) + c_2\, e^{x} + \frac{1}{2}e^{2x}(x-1)$$

Ex: The oscillation of a string under the gravitational force

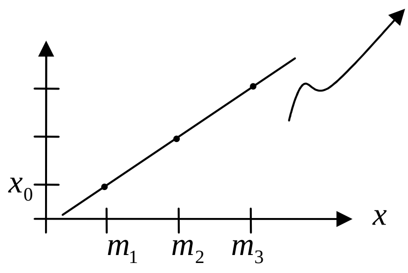


At balance,

$$\left|\vec{\mathbf{F}}_{spring}\right| = \left|\vec{\mathbf{F}}_{gravity}\right| = mg$$

One can vary m to vary the distance x

Question: What is the relation between $\vec{\mathbf{F}}_{\mathbf{g}}$ and how far will the string be stretched?



$x = x_0 + K_m$, where K : proportionality const.

$$m = \frac{\left|\vec{\mathbf{F}}_{\mathbf{gravity}}\right|}{g} = \frac{\left|\vec{\mathbf{F}}_{\mathbf{spring}}\right|}{g}$$

$$x = x_0 + \frac{K}{g}F_{spring}$$

$$\therefore F_{spring} = \frac{g}{k}(x - x_0)$$

$$= k(x - x_0), \quad k = \text{spring constant}$$

Newton's Law: $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$

$$mg - k(x - x_0) = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}(x - x_0) = g$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = g + \frac{kx_0}{m}$$

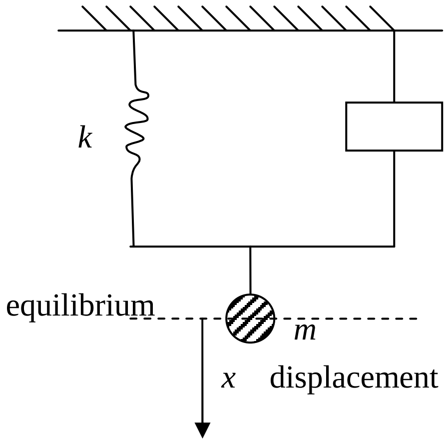
Homogeneous solution: $x_1 = \sin\left(\sqrt{\frac{k}{m}}\, t\right)$, $x_2 = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

Particular solution: $x_p(t) = x_0 + \frac{mg}{k}$ ← the equilibrium point of the mass m hanging on the spring

Thus the general solution is

$$x = c_1 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + \left(x_0 + \frac{mg}{k}\right)$$

Add a "damping" term to the spring system



$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$

or $m\frac{d^2X}{dt^2} = -kX - c\frac{dX}{dt}$

Note that X is measured from the equilibrium point, the effect of gravity ("mg" term) is eliminate.

$\left[X = x - \left(x_0 + \frac{mg}{k}\right)\right]$

Governing equation for the spring system with the damping term:

$$m\ddot{X} + c\dot{X} + kX = 0 \quad \text{,where } c > 0,\ \ m > 0,\ \ k > 0 \qquad (5.19)$$

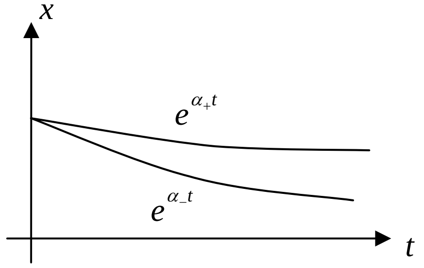
Where X = displacement from the equilibrium point

Characteristic equation: Substituting $X = e^{\alpha t}$ into $m\ddot{X} + c\dot{X} + kX = 0$

$$\Rightarrow m\alpha^2 + c\alpha + k = 0$$

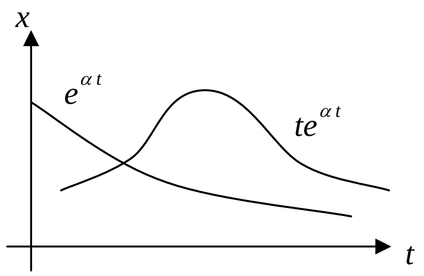
$$\Rightarrow \alpha_{\pm} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m} \qquad (5.20)$$

Case Ⅰ) $c^2 > 4mk$: $\alpha_+ < 0, \ \ \alpha_- < 0$

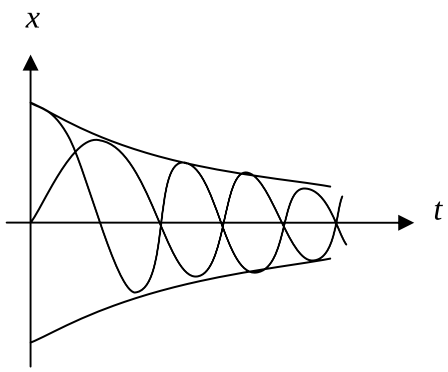


OVER DAMPED

Case Ⅱ) $c^2 = 4mk$: $\alpha_+ = \alpha_- = \dfrac{-c}{2m}$



Case Ⅲ) $c^2 < 4mk$: solution as $e^{rt}\left(\sin\beta t, \ \cos\beta t\right)$



For c=0, i.e., no damping effect and replace X by x in (5.19)

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (*)$$

Multiply (*) by $\dot{x}$, i.e., applying "energy integral"

$$\Rightarrow m\ddot{x}\dot{x} + kx\dot{x} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2\right) = 0$$

$$\Rightarrow \underbrace{\frac{m}{2}\dot{x}^2}_{\text{kinetic energy}} + \underbrace{\frac{k}{2}x^2}_{\text{potential energy stored in the spring}} = \varepsilon_0 = \text{constant} \qquad (5.21)$$

Thus, $\frac{m}{2}\left(\frac{dx}{dt}\right)^2 = \varepsilon_0 - \frac{k}{2}x^2$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2\varepsilon_0}{m} - \frac{k}{m}x^2}$$

Using the "separation of variable" method

$$\int \frac{dx}{\sqrt{\frac{2\varepsilon_0}{m} - \frac{k}{m}x^2}} = \int dt \qquad \left[\text{Note that : integration formula} \quad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \sqrt{\frac{m}{k}} \sin^{-1}\left(\frac{x}{\sqrt{\frac{2\varepsilon_0}{k}}}\right) = t + c$$

$$\Rightarrow x(t) = \sqrt{\frac{2\varepsilon_0}{k}} \sin\left(\sqrt{\frac{k}{m}}\, t + \phi\right), \ \phi = c\sqrt{\frac{k}{m}} \qquad (5.22)$$

$$\left[\text{OR} \quad x(t) = c_1 \sin\left(\sqrt{\frac{k}{m}}\, t\right) + c_2 \cos\left(\sqrt{\frac{k}{m}}\, t\right)\right]$$

Ex: Solve this ODE, $\ddot{x} + G(x) = 0$, where $\ddot{x} = \frac{d^2x}{dt^2}$

Sol: Using the energy integral method

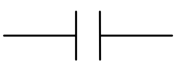
$$\Rightarrow \dot{x}[\ddot{x} + G(x)] = 0$$

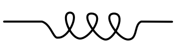
$$\ddot{x}\dot{x} + G(x)\dot{x} = 0$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}(\dot{x})^2 + \underbrace{\int_0^x G(\tilde{x})d\tilde{x}}_{\text{assume } V(x)}\right] = 0$$

$$\Rightarrow \frac{1}{2}(\dot{x})^2 + V(x) = \text{constant} \ \text{, where } V(x) = \int_0^x G(\tilde{x})d\tilde{x} \ \text{ or } \ \frac{dV(x)}{dx} = G(x)$$

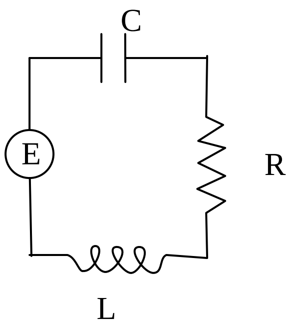
<u>Application in Circuit Problem</u>:

 capacitor, C in farad

 inductor, L in henry

 resistor, R in ohm

For this electrical circuit,



The capacitor stores charge

$$V_C = \frac{Q}{C} \text{ , } Q \text{ is Columb (charge)}$$

The resistor resists current

$$V_R = RI \text{ , } I \text{ in Ampery}$$

The inductor resists charge in the current

$$V_L = L\frac{dI}{dt} \text{ , and } I = \frac{dQ}{dt}$$

Kirchhoff's Law: In a closed circuit, the impressed voltage equals the sum of the voltage drops in the rest of the circuit.

$$\frac{Q}{C} + L\frac{dI}{dt} + RI = E(t) \text{ , and } I = \frac{dQ}{dt}$$

$$\text{ODE : } L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad (*) \quad \text{, where } L > 0,\ R > 0,\ C > 0$$

For an oscillatory applied voltage, $E(t) = E_0 \cos(\omega_0 t)$

The above ODE becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E_0 \cos(\omega_0 t) \qquad (5.23)$$

Assume the particular solution $Q_p(t)$ to (5.23) as

$$Q_p(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t) \qquad (5.24)$$

Substitution (5.24) into (5.23) to solve A and B, then we can get

$$A = \frac{E_0\omega_0 R}{\left(\frac{1}{C} - \omega_0{}^2 L\right)^2 + \omega_0{}^2 R^2}, \quad B = \frac{E_0\left(\frac{1}{C} - \omega_0{}^2 L\right)}{\left(\frac{1}{C} - \omega_0{}^2 L\right)^2 + \omega_0{}^2 R^2} \qquad (5.25)$$

Note that if $R \neq 0$, the homogeneous solution $Q_H(t)$ will be damped to zero because of electrical resistance (R)

Ⅰ) Suppose $R \to 0$:

The particular solution will approach $Q_p(t) = \dfrac{E_0}{\left(\dfrac{1}{C} - \omega_0{}^2 L\right)} \cos(\omega_0 t)$

The initial conditions are: $Q(0) = 0$, $\dot{Q}(0) = I(0) = 0$

The ODE (5.23) can be rewritten as

$$L\ddot{Q} + \frac{Q}{C} = E_0 \cos(\omega_0 t) \qquad (5.26)$$

Homogeneous solutions to Eq. (5.26) are

$$\left.\begin{aligned} Q_1(t) &= \sin\left(\sqrt{\frac{1}{LC}}\, t\right) \\ Q_2(t) &= \cos\left(\sqrt{\frac{1}{LC}}\, t\right) \end{aligned}\right\} \qquad (5.27)$$

$$\text{Define}: \sqrt{\frac{1}{LC}} \equiv \omega$$

So the particular solution can be written as

$$Q_p(t) = \frac{E_0}{L\left(\omega^2 - \omega_0{}^2\right)} \cos(\omega_0 t) \qquad (5.28)$$

Ⅱ) Suppose that ω is closed to ω_0:

The general solution to the ODE in Eq.(5.23) is

$$Q(t) = c_1 \sin \omega t + c_2 \cos \omega t + \frac{E_0}{L\left(\omega^2 - \omega_0^2\right)} \cos(\omega_0 t)$$

check with ICs: $Q(0) = 0$, $\dot{Q}(0) = 0$

$$\Rightarrow c_1 = 0 \quad , c_2 = \frac{-E_0}{L\left(\omega^2 - \omega_0^2\right)}$$

$$\therefore \text{solution is} \quad Q(t) = \frac{E_0}{L\left(\omega^2 - \omega_0^2\right)} \left[\cos(\omega_0 t) - \cos(\omega t)\right], \ \omega \equiv \frac{1}{\sqrt{LC}} \qquad (5.29)$$

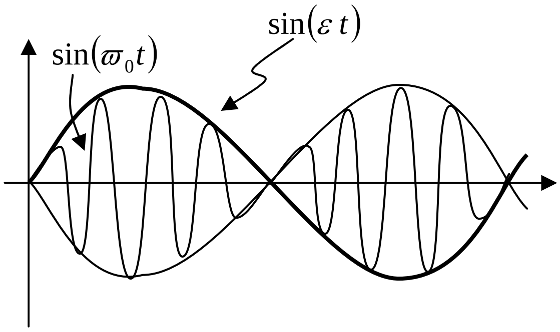
Since ω is closed to ω_0

$$\Rightarrow \frac{E}{L\left(\omega^2 - \omega_0^2\right)} \quad \text{is very large}$$

$$\cos(\omega_0 t) - \cos(\omega t) = -2 \sin\left(\frac{\omega_0 + \omega}{2} t\right) \sin\left(\frac{\omega_0 - \omega}{2} t\right)$$

$$\approx -2 \sin(\omega_0 t) \sin(\varepsilon t) \quad , \text{where } \varepsilon = \frac{\omega_0 - \omega}{2} \text{ is small}$$

Thus, $$Q(t) \approx \frac{-2E_0}{L(\omega^2 - \omega_0^2)} \sin(\omega_0 t)\sin(\varepsilon t) \qquad (5.30)$$



Amplitude Modulation
(A. M.)

III) Suppose that $L \to 0$:

ODE: $$R\dot{Q} + \frac{1}{C}Q = E_0 \cos(\omega_0 t)$$

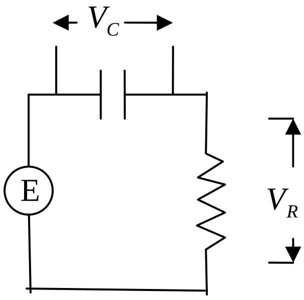
$$\Rightarrow Q_p(t) = \frac{E_0}{\left(\frac{1}{C}\right)^2 + \omega_0^2 R^2}\left(\omega_0 R \sin\omega_0 t + \frac{1}{C}\cos\omega_0 t\right)$$

$$= A\sin(\omega_0 t + \phi) \qquad (5.31)$$

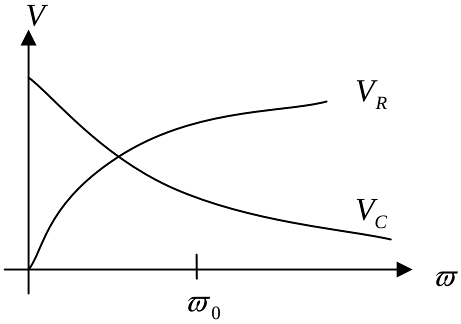
where $$A = \frac{E_0}{\sqrt{\left(\frac{1}{c}\right)^2 + \omega_0^2 R^2}}, \quad \phi = \cos^{-1}\left(\frac{\omega_0 R}{\sqrt{\left(\frac{1}{c}\right)^2 + \omega_0^2 R^2}}\right)$$

$$\Rightarrow V_C = \frac{|Q_p|}{C} = \frac{E_0}{\sqrt{1 + R^2\omega_0^2 C^2}}$$

$$V_R = RI = R\frac{dQ}{dt} = \frac{R\omega_0 E_0}{\sqrt{\left(\frac{1}{C}\right)^2 + R^2\omega_0^2}}$$



$$V_C = V_C(\omega_0)$$
$$V_R = V_R(\omega_0)$$



If $\omega >> \omega_0 \Rightarrow V_R >> V_C$
If $\omega << \omega_0 \Rightarrow V_C >> V_R$

- Maxwell's Equations in Electromagnetics:

$$\begin{cases} \vec{\nabla}\cdot\vec{\mathbf{D}} = \rho \\ \vec{\nabla}\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + \dfrac{\partial\vec{\mathbf{D}}}{\partial t} \\ \vec{\nabla}\times\vec{\mathbf{E}} + \dfrac{\partial\vec{\mathbf{B}}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{\mathbf{B}} = 0 \end{cases}$$

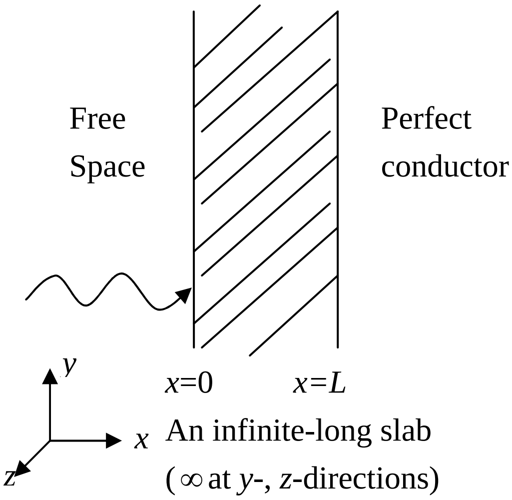
where $\vec{\mathbf{D}} = \varepsilon\,\vec{\mathbf{E}}$, $\vec{\mathbf{J}} = \sigma\,\vec{\mathbf{E}}$, $\vec{\mathbf{B}} = \mu\,\vec{\mathbf{H}}$

ε = permittivity or dielectric tensor
μ = magnetic permeability
σ = conductivity (1/resistence)
$\vec{\mathbf{E}}$ = electric field, $\vec{\mathbf{H}}$ = magnetic field
$\vec{\mathbf{B}}$ = magnetic induction, $\vec{\mathbf{D}}$ = electric displacement
$\vec{\mathbf{J}}$ = current density, ρ = charge density

Review: $\vec{\nabla}\cdot\vec{\mathbf{V}} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$, $\vec{V} = (u, v, w)$

$$\vec{\nabla}\times\vec{\mathbf{V}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Physical Situation:



An infinite-long slab
(∞ at y-, z-directions)

Consider the following conditions:

(1) A light wave traveling in x direction

light speed, $c = \dfrac{\omega}{k_0}$

Incoming wave: $e^{i(k_0 x - \omega t)}$

$= \cos(k_0 x - \omega t) + i \sin(k_0 x - \omega t)$

(2) $\rho = 0 \Rightarrow$ charge density is zero

(3) ∞-long slab at y-, z-direction

$\Rightarrow$ No y-, z-dependence

Thus, for the Maxwell's equations

$$\begin{cases} \vec{\nabla} \times \vec{\mathbf{E}} + \dfrac{\partial \vec{\mathbf{B}}}{\partial t} = 0 \\ \vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \dfrac{\partial \vec{\mathbf{D}}}{\partial t} \end{cases} \quad \Rightarrow \text{only x-component equation}$$

$$\because \frac{\partial}{\partial t}\left[e^{i(k_0 x - \omega t)}\right] = -i\varpi\, e^{i(k_0 x - \omega t)}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \equiv -i\omega}$$

$$\therefore \begin{cases} \vec{\nabla} \times \vec{\mathbf{E}} - i\omega \vec{\mathbf{B}} = 0 & (5.32) \\ \vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}} - i\omega \vec{\mathbf{D}} & (5.33) \end{cases} \quad \left(\begin{array}{l} \text{Recall}: \vec{\mathbf{H}} = \dfrac{1}{\mu}\vec{\mathbf{B}}, \vec{\mathbf{J}} = \sigma \vec{\mathbf{E}} \\ \vec{\mathbf{D}} = \varepsilon \vec{\mathbf{E}} \end{array} \right)$$

$\because$ $\vec{\mathbf{E}}, \vec{\mathbf{H}} \perp$ each other (electrical field is normal to magnetic field)

let us assume $\begin{cases} \vec{\mathbf{E}} = E(x)\hat{\mathbf{j}} \\ \vec{\mathbf{H}} = H(x)\hat{\mathbf{k}} \end{cases}$

$$\therefore \vec{\nabla} \times \vec{\mathbf{E}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E(x) & 0 \end{vmatrix} = \frac{\partial E}{\partial x}\hat{\mathbf{k}} = E'(x)\hat{\mathbf{k}}$$

$$\vec{\nabla} \times \vec{\mathbf{H}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & H(x) \end{vmatrix} = -\frac{\partial H}{\partial x}\hat{\mathbf{j}} = -H'(x)\hat{\mathbf{j}}$$

So (5.32) $\Rightarrow \vec{\nabla} \times \vec{\mathbf{E}} - i\omega \vec{\mathbf{B}} = E'(x)\hat{\mathbf{k}} - i\omega\mu H(x)\hat{\mathbf{k}} = \hat{\mathbf{k}}\left[E'(x) - i\omega\mu H(x)\right] = 0$

$$\Rightarrow E'(x) - i\varpi\mu H(x) = 0 \qquad (5.34)$$

LHS of (5.33) : $\vec{\nabla} \times \vec{\mathbf{H}} = -H'(x)\hat{\mathbf{j}}$

RHS of (5.33) : $\vec{\mathbf{J}} - i\omega\vec{\mathbf{D}} = \sigma E(x)\hat{\mathbf{j}} - i\omega\varepsilon E(x)\hat{\mathbf{j}}$

LHS = RHS $\Rightarrow -H'(x) = (\sigma - i\omega\varepsilon)E(x) \qquad (5.35)$

Assume: $\sigma, \varepsilon, \mu =$ constant through the slab

$$\frac{d}{dx}(5.34) \Rightarrow E''(x) - i\omega\mu H'(x) = 0$$

$$H'(x) = (i\omega\varepsilon - \sigma)E(x) \qquad (5.35)$$

$$\Rightarrow E''(x) + \mu\left(\omega^2\varepsilon + i\omega\sigma\right)E(x) = 0 \qquad (5.36)$$

Eq. (5.36) is an 2[nd]-order ODE with const. coefficients (complex variables)

Assume: $E(x) = e^{\alpha x}$, substituting to (5.36)

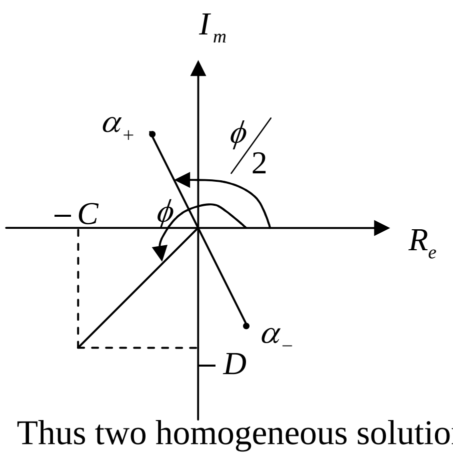
$$\Rightarrow \alpha^2 + \mu\left(\omega^2\varepsilon + i\omega\sigma\right) = 0$$

rewrite as

$$\alpha^2 + (C + iD) = 0 \quad \text{, where } C = \mu\varepsilon\omega^2 > 0 \text{ , D} = \mu\omega\sigma > 0$$

$$\Rightarrow \alpha_{\pm} = \pm\sqrt{-C - iD}$$

Complex Plane



Let $-C - iD = R\,e^{i\phi}$ (complex variable)

Then $\pm\sqrt{R\,e^{i\phi}} = \begin{cases} \sqrt{R}\,e^{i\frac{\phi}{2}} \\ \sqrt{R}\,e^{i\left(\pi + \phi/2\right)} \end{cases}$

Two roots $\left(e^{\alpha_+ x}, e^{\alpha_- x}\right)$ where $\begin{matrix} \alpha_+ = -A + iB \\ \alpha_- = A - iB \end{matrix}$

Thus two homogeneous solutions to Eq.(5.36) are

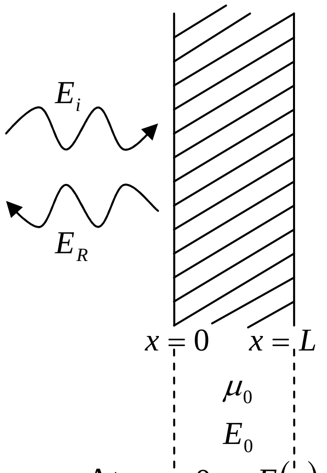
$$\begin{cases} E_1(x) = e^{-Ax}e^{iBx} \\ E_2(x) = e^{Ax}e^{-iBx} \end{cases} \qquad (5.37)$$

Then, the general solution to Eq.(5.36) is

$E(x) = c_1E_1(x) + c_2E_2(x)$, where $E_1(x), E_2(x)$ are given in (5.37)

c_1, c_2 are determined by BCs

Assume:



Perfect Conductor

Incoming EM Wave: $E_i e^{i(k_0 x-\omega t)}$

Reflected EM Wave: $E_R e^{i(-k_0 x-\omega t)}$

[Note"–": opposite phase]

Let $\underbrace{\dfrac{E_R}{E_i}}_{\text{Reflectivity}} = R \Rightarrow E_R = RE_i$

At $x<0$ $\quad E(x)=E_i\left(e^{ik_0x}+R\,e^{-ik_0x}\right)e^{-i\omega t}$ (5.38)

At $x=0$ $\quad E(0)=E_i(1+R)e^{-i\omega t}$ (5.39)

Inside the slab ($0\le x\le L$), according to the Eq.(5.34)

$$H(x)=\frac{1}{i\omega\mu}E'(x)=\frac{-i}{\omega\mu}E'(x) \qquad (5.40)$$

At $x<0$

$$H(x)=\frac{-i}{\omega\mu_0}E'(x)=\frac{-i}{\omega\mu_0}E_i\left(ik_0e^{ik_0x}-R\,ik_0e^{-ik_0x}\right)e^{-i\omega t} \qquad (5.41)$$

BC (1): Continuity of $H(x)$ at $x=0$:

$$H(0)\underset{\uparrow\,(5.41)}{=}\frac{-iE_i}{\omega\mu_0}ik_0(1-R)e^{-i\omega t}\underset{\uparrow\,(5.40)}{=}\frac{-i}{\omega\mu}E'(0)$$

i.e., $\dfrac{E_ik_0}{\omega\mu_0}(1-R)e^{-i\varpi t}=\dfrac{-i}{\omega\mu}E'(0)$

$$\Rightarrow \frac{E_ik_0}{\omega\mu_0}(1+R-2R)e^{-i\omega t}=\frac{-i}{\omega\mu}E'(0)$$

$$\Rightarrow \frac{\mu}{-i}\left\{\underbrace{E_i\frac{k_0}{\mu_0}(1+R)e^{-i\omega t}}_{\frac{k_0}{\mu_0}E(0)\ (5.39)}-2R\frac{E_ik_0}{\mu_0}e^{-i\omega t}=\frac{-i}{\mu}E'(0)\right\}$$

$$\Rightarrow E'(0)-i\frac{\mu}{\mu_0}k_0E(0)=-i2R\frac{E_ik_0\mu}{\mu_0}e^{-i\omega t} \qquad (5.42)$$

(A mixed boundary condition)

BC (2): $E(L)=0$ (5.43)

($\because$ the material is a perfect conductor for $x>L \Rightarrow E(x)=0$ at $x>L$)

Substitution (5.37) into (5.42) and (5.43), we can find c_1 and c_2
Thus the solution is

$$E(x) = c_1 E_1(x) + c_2 E_2(x)$$

where $E_1(x) = e^{-Ax} e^{iBx}$, $E_2(x) = e^{Ax} e^{-iBx}$