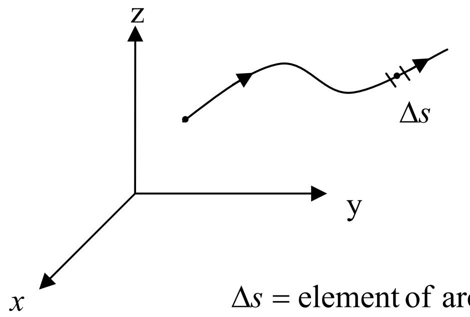
Chapter 4 Line Integrals

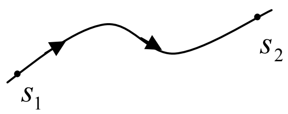


$$\int_c f(x,y,z)ds = \lim_{\substack{\Delta s\to 0\\ N\to\infty}} \sum_{i=1}^{N} f(x_i, y_i, z_i)\Delta s_i$$

$\Delta s =$ element of arc length

Specify the curve C:

$$C: \begin{cases} x = x(s) \\ y = y(s) \\ z = z(s) \end{cases}$$, where $s \equiv$ arc length along curve C



$$\int_c f(x,y,z)ds = \int_{s_2}^{s_1} f(x(s), y(s), z(s))\, ds$$

arc length $= s_2 - s_1$

If

$$\text{curve } C: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

t : arc any parameter along curve C

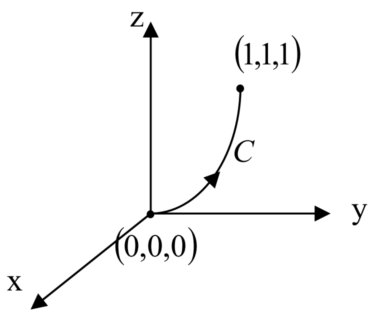
$t \in (t_1, t_2)$ along C

Thus

$$\int_c f(x,y,z)ds = \int_{s_2}^{s_1} f(x(s), y(s), z(s))\left(\frac{ds}{dt}\right) dt$$

$$\text{and } \frac{ds}{ds} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad (4.1)$$

Ex:



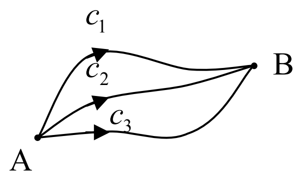
For $f(x,y,z) = x + y + z$

$$C: x = x(t) = t, \quad y = y(t) = t, \quad t \in (0,1), \quad z = z(t) = t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{3}, \therefore ds = \sqrt{3}dt$$

$$\therefore \int_c f\ ds = \int_0^1 (3t)\sqrt{3}dt = 3\sqrt{3}\left[\frac{t^2}{2}\right]_0^1 = \frac{3}{2}\sqrt{3}$$

In general, consider the line integral $\int_c f(x,y,z)\,ds$

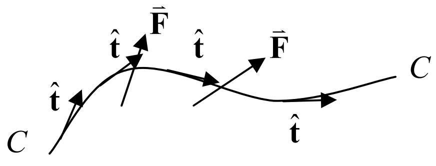


$$\int_{c_1} f(x,y,z)\,ds \neq \int_{c_2} f(x,y,z)\,ds \neq \int_{c_3} f(x,y,z)\,ds$$

if $c_1 \neq c_2 \neq c_3$

What condition must f satisfy if $\int_{c_1} f\ ds = \int_{c_2} f\ ds$?

(i.e., the line integral is path independent!)



Consider $\int_c \vec{\mathbf{F}}(x,y,z)\cdot \hat{\mathbf{t}}\,ds$

where $\hat{\mathbf{t}}$ = unit tangent vector along curve C

for curve $C: \begin{cases} x = x(s) \\ y = y(s), \ s: \text{arc length} \\ z = z(s) \end{cases}$

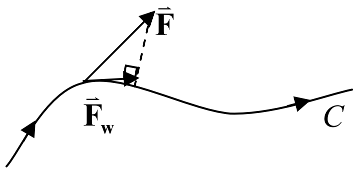
then $\hat{\mathbf{t}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$ (unit tangent vector)

If Curve $C:\begin{cases} x = x(t) \\ y = y(t), \\ z = z(t) \end{cases}$ t : not arc length, any parameter

then $\hat{\mathbf{t}} = \dfrac{\dfrac{dx}{dt}\hat{\mathbf{i}} + \dfrac{dy}{dt}\hat{\mathbf{j}} + \dfrac{dz}{dt}\hat{\mathbf{k}}}{\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2}}$ (unit tangent vector)

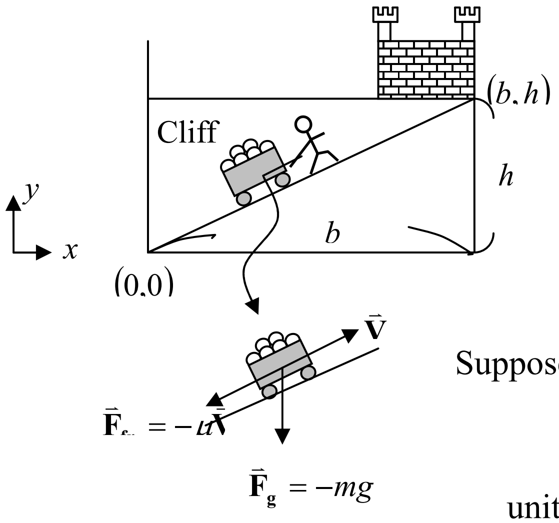
Definition of "Work":

$$W_{\vec{\mathbf{F}}} \equiv \int_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds \qquad (4.2)$$



The component of force $\mathbf{F}$ which dose work acts only along the direction of motion.

Ex:



$$W_{peasant} = -W_{FORCE} = -\int_0^s \left(\mathbf{F_g} + \mathbf{F_{fr}}\right) \cdot \hat{\mathbf{t}}\, ds$$

Curve $C:\begin{cases} x = t \\ y = \dfrac{h}{b}t \end{cases}$

Suppose this peasant moves at a constant speed

$|\mathbf{V}| = u$, or $\mathbf{V} = u\hat{\mathbf{t}}$

unit tangent vector is

$$\hat{\mathbf{t}} = \frac{\left(\dfrac{dx}{dt}\right)\hat{\mathbf{i}} + \left(\dfrac{dy}{dt}\right)\hat{\mathbf{j}}}{\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}} = \frac{\hat{\mathbf{i}} + \dfrac{h}{b}\hat{\mathbf{j}}}{\sqrt{1 + \left(\dfrac{h}{b}\right)^2}}$$

Thus the work done by the peasant is

$$W_p = -\int_c \left(-mg\hat{\mathbf{j}} - \mu\mathbf{v}\right)\cdot\hat{\mathbf{t}}\sqrt{1+\left(\frac{h}{b}\right)^2}\ dt$$

$$= \int_0^b \left[mg\left(\hat{\mathbf{j}}\cdot\hat{\mathbf{t}}\right) + \mu\, u\right]\sqrt{1+\left(\frac{h}{b}\right)^2}\ dt$$

$$= \int_0^b mg\left[\hat{\mathbf{j}}\cdot\frac{\hat{\mathbf{i}}+\frac{h}{b}\hat{\mathbf{j}}}{\sqrt{1+\left(\frac{h}{b}\right)^2}}\right]\sqrt{1+\left(\frac{h}{b}\right)^2}\,dt + b\mu\, u\sqrt{1+\left(\frac{h}{b}\right)^2}$$

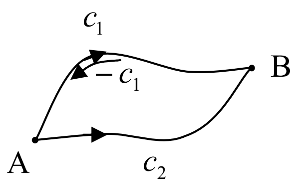
$$= \int_0^b mg\frac{h}{b}\,dt + b\mu\, u\sqrt{1+\left(\frac{h}{b}\right)^2}$$

$$= \underline{mgh} + \underline{b\mu\, u\sqrt{1+\left(\frac{h}{b}\right)^2}}$$

mgh: the work done against the gravitational force

$b\mu\, u\sqrt{1+\left(\frac{h}{b}\right)^2}$: the work done against the frictional force

Consider the path independence of the line integral



If $\int_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds$ (*)

since $\int_{-C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\int_{C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds$

$(*) \Rightarrow -\int_{-C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds$

$\Rightarrow 0 = \int_{-C_1}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds + \int_{C_2}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds$

$\Rightarrow \int_{C_2+(-C_1)}\mathbf{F}\cdot\hat{\mathbf{t}}\,ds = 0$

$$\boxed{\therefore\ \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = 0} \qquad (4.3)$$

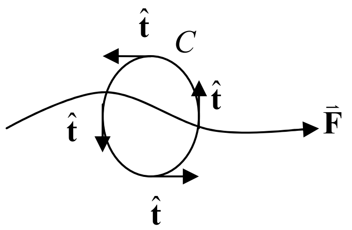
where C is the closed contour

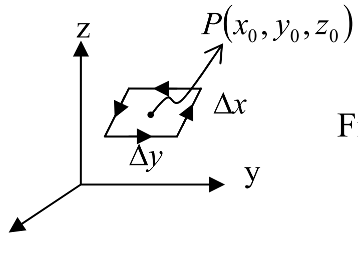
(as combined by path C_2 and $-C_1$)

Note that (4.3) is true only when $\nabla\times\mathbf{F} = \vec{0}$ $(\text{curl}\,\mathbf{F} = 0)$

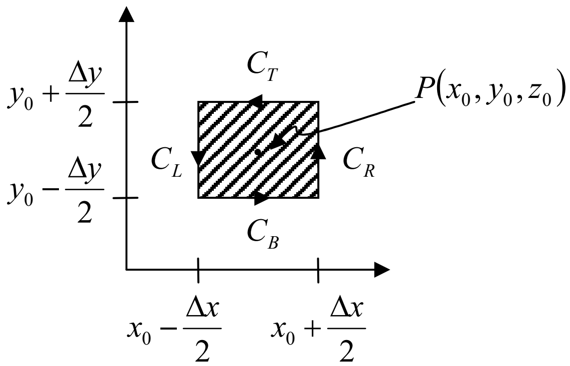
- Definition of curl $\mathbf{F}$:

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds \equiv \text{circulation of } \mathbf{F} \qquad (4.4)$$





Find $\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = ?$



$$\int_{C_B} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{C_B} F_x\,ds$$

$$= \int_{x_0-\frac{\Delta x}{2}}^{x_0+\frac{\Delta x}{2}} F_x\left(x, y_0-\frac{\Delta y}{2}, z_0\right)dx$$

$$\approx F_x\left(x_0, y_0-\frac{\Delta y}{2}, z_0\right)\Delta x$$

Similarly $\int_{C_T} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{C_T} -F_x\left(x, y_0+\frac{\Delta y}{2}, z_0\right)ds \approx -F_x\left(x_0, y_0+\frac{\Delta y}{2}, z_0\right)\Delta x$

Thus $\int_{C_B+C_T} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \left[F_x\left(x_0, y_0-\frac{\Delta y}{2}, z_0\right) - F_x\left(x_0, y_0+\frac{\Delta y}{2}, z_0\right)\right]\Delta x$

and $\Delta A = \Delta x \Delta y$

$$\therefore \frac{1}{\Delta A}\int_{C_B+C_T} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{F_x\left(x_0, y_0-\frac{\Delta y}{2}, z_0\right) - F_x\left(x_0, y_0+\frac{\Delta y}{2}, z_0\right)}{\Delta y}$$

As $\Delta A \to 0$, then $\lim_{\Delta A\to 0}\frac{1}{\Delta A}\int_{C_B+C_T} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = -\frac{\partial F_x}{\partial y}(x_0, y_0, z_0)$

Similary, $\lim_{\Delta A \to 0} \frac{1}{\Delta A} \int_{C_L + C_R} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = -\frac{\partial F_y}{\partial x}(x_0, y_0, z_0)$

$\therefore \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$ ($\hat{\mathbf{k}}$ - component of curl $\mathbf{F}$)

Similarly, $\hat{\mathbf{i}}$ - component of curl $\mathbf{F} = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$

$\hat{\mathbf{j}}$ - component of curl $\mathbf{F} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$

Note that $\nabla = \text{del notation} \equiv \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$

$$\nabla \cdot \mathbf{F} = \text{div}\, \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \text{ for } \mathbf{F} = F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}} + F_z \hat{\mathbf{k}}$$

$$\text{curl}\, \mathbf{F} = \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

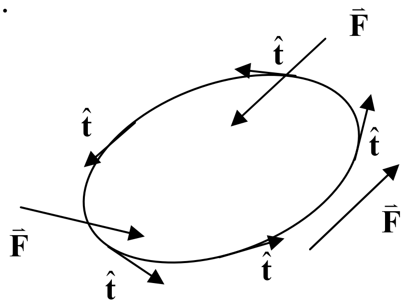
$$= \hat{\mathbf{i}} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \qquad (4.5)$$

Note that

Normal component of curl $\mathbf{F} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$

$$\therefore \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}} \qquad (4.6)$$

Ex:

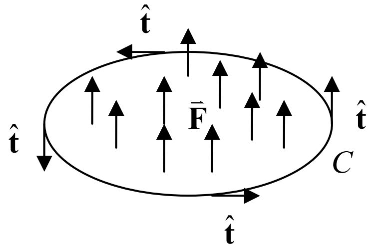


$\because \oint_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds > 0$

$\therefore \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \text{curl}\, \mathbf{F} \cdot \hat{\mathbf{n}} > 0$

Note: curl $\mathbf{F}$ can be expressed as a measurement of rotation if $\mathbf{F}$ is velocity vector.

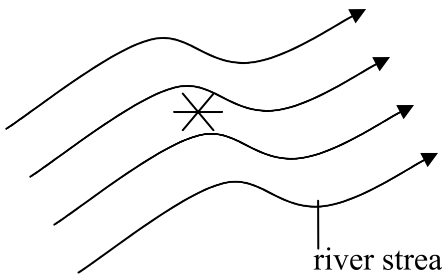
Ex:



$\because$ Along the curve C, every tangent vector $\hat{\mathbf{t}}_1$ has its opposite tangent vector $\left(\hat{\mathbf{t}}_2 = -\hat{\mathbf{t}}_1\right)$

$\therefore \oint_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = 0 \Rightarrow \text{curl}\,\mathbf{F} = \vec{\mathbf{0}}$

Ex:



The wheel ($\times$) will spin around an axis

$\Rightarrow$ the axis direction is the direction of curl $\mathbf{F}$

Ex: For solid-body rotation, $\mathbf{F}$ is the velocity vector,

$\Rightarrow$ $\text{curl}\,\mathbf{F} = 2\boldsymbol{\omega}$, where $|\boldsymbol{\omega}| = \dfrac{d\theta}{dt}$, the angular velocity vector.

- For the general orthogonal curvilinear coordinate (u, v, w)

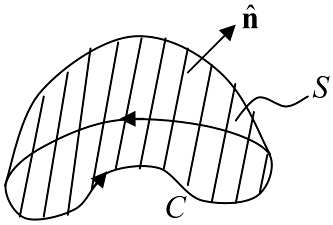
$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_1 F_u & h_2 F_v & h_3 F_w \end{vmatrix}$$

$$= \frac{1}{h_2 h_3}\hat{\mathbf{e}}_1\left[\frac{\partial}{\partial v}(h_3 F_w) - \frac{\partial}{\partial w}(h_2 F_v)\right] + \frac{1}{h_1 h_3}\hat{\mathbf{e}}_2\left[\frac{\partial}{\partial w}(h_1 F_u) - \frac{\partial}{\partial u}(h_3 F_w)\right]$$

$$+ \frac{1}{h_1 h_2}\hat{\mathbf{e}}_3\left[\frac{\partial}{\partial u}(h_2 F_v) - \frac{\partial}{\partial v}(h_1 F_u)\right] \qquad (4.7)$$

Note that $\vec{\nabla} \cdot \left(\vec{\nabla} \times \mathbf{F}\right) \equiv 0$. , i.e., curl $\mathbf{F}$ has no divergence !

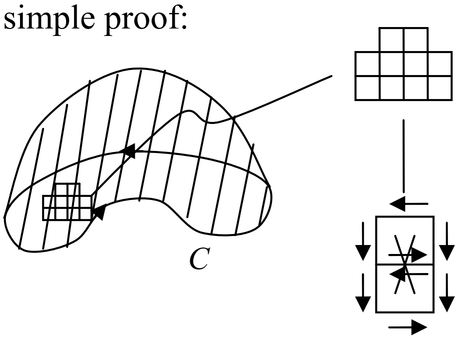
- Stokes' theorem



$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \iint_S \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\,dA \tag{4.8}$$

S : any surface capped onto curve C

A simple proof:

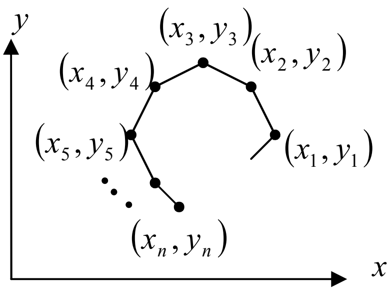


$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \sum_{C_i}\oint_{C_i} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$

then the only integral not cancelled out is the integral along curve C

thus
$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \sum_{C_i}\oint \mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$
$$= \sum_{C_i}\left(\frac{1}{\Delta A_i}\oint_{C_i} \mathbf{F}\cdot\hat{\mathbf{t}}\,ds\right)\Delta A_i$$
$$= \sum_{C_i}\left(\operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\right)_i \Delta A_i$$
$$= \iint_S \left(\operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\right) dA$$

Example: Using the Stokes' theorem to define the area of a polygon with N sides.



$$\because \iint_S dA = A$$

By the Stokes' theorem

$$\iint_S \left(\operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\right) dA = \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$

Let us choose $A = \iint_S \left(\operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{n}}\right) dA$

$$\Rightarrow \text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}} = 1$$

$$\because \hat{\mathbf{n}} = \hat{\mathbf{k}} \quad \therefore (\text{curl}\,\mathbf{F})_z = 1$$

$$\Rightarrow \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1$$

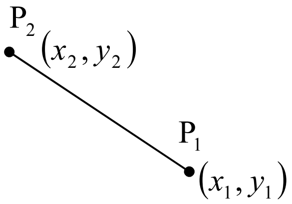
For simplicity, let us take

$$\begin{cases} F_y = x \\ F_x = 0 \Rightarrow \mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} = x\hat{\mathbf{j}} \\ F_z = 0 \end{cases}$$

$\therefore$

$$A = \iint_S (\text{curl}\,\mathbf{F}\cdot\hat{\mathbf{n}})\,dA = \oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds$$ i.e., Stoke's theorem

Consider the line as



$$\hat{\mathbf{t}} \text{ along this line} = \frac{(x_2 - x_1)\hat{\mathbf{i}} + (y_2 - y_1)\hat{\mathbf{j}}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

The parametric form along the line:

$$\begin{cases} x = x_1 + (x_2 - x_1)t \\ y = y_1 + (y_2 - y_1)t \end{cases}, \quad 0 \le t \le 1$$

$$\text{then } ds = \left(\frac{ds}{dt}\right)dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,dt$$

$$\therefore \int_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \int_{(x_1,y_1)\to(x_2,y_2)} \left(x\,\hat{\mathbf{j}}\right)\cdot\frac{\left[(x_2-x_1)\hat{\mathbf{i}}+(y_2-y_1)\hat{\mathbf{j}}\right]}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\,dt$$

$$= \int_{(x_1,y_1)\to(x_2,y_2)} \left(x\,\hat{\mathbf{j}}\right)\cdot\left[(x_2-x_1)\,\hat{\mathbf{i}}+(y_2-y_1)\,\hat{\mathbf{j}}\right]dt$$

$$= \int_0^1 \left[x_1+(x_2-x_1)t\right](y_2-y_1)\,dt$$

$$= x_1(y_2-y_1)t\Big|_0^1 + (x_2-x_1)(y_2-y_1)\frac{t^2}{2}\Big|_0^1$$

$$= x_1(y_2-y_1)+\frac{1}{2}(x_2-x_1)(y_2-y_1)$$

$$= \frac{1}{2}(y_2-y_1)(x_2+x_1)$$

$\therefore$ For $P_1 \to P_2$, $\quad \int = \frac{1}{2}\left(x_2y_2 - y_1x_2 + y_2x_1 - y_1x_1\right)$ ①

$P_2 \to P_3$, $\quad \int = \frac{1}{2}\left(x_3y_3 - y_2x_3 + y_3x_2 - y_2x_2\right)$ ② ①

$P_3 \to P_4$, $\quad \int = \frac{1}{2}\left(x_4y_4 - y_3x_4 + y_4x_3 - y_3x_3\right)$ ③ ② ③

$\vdots \qquad\qquad \vdots$

$+)\quad P_N \to P_1$, $\quad \int = \frac{1}{2}\left(x_1y_1 - y_Nx_1 + y_1x_N - y_Nx_N\right)$ N−1

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{t}}\,ds = \frac{1}{2}\sum_{i=1}^{N-1}\left[x_iy_{i+1}-y_ix_{i+1}\right]+\frac{1}{2}\left(x_Ny_1-y_Nx_1\right)$$

$$= \frac{1}{2}\oint\left(x\,dy - y\,dx\right)$$

$$\vec{\nabla}\cdot\mathbf{G}=0 \Leftrightarrow \mathbf{G}=\text{curl}\,\mathbf{H} \tag{4.9}$$

$$\because\ \nabla\cdot\left(\nabla\times\mathbf{H}\right)=\nabla\cdot\left(\text{curl}\,\mathbf{H}\right)=0$$

$$\vec{\nabla}\times\mathbf{F}=0 \Leftrightarrow \mathbf{F}=\text{grad}\,\phi(x,y,z)=\vec{\nabla}\phi \tag{4.10}$$

where $\phi(x,y,z)$ is a scalar function

$$\vec{\nabla}\phi = \text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}}+\frac{\partial\phi}{\partial y}\hat{\mathbf{j}}+\frac{\partial\phi}{\partial z}\hat{\mathbf{k}}$$

$$\text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right)\hat{\mathbf{i}} + \left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right)\hat{\mathbf{j}} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right)\hat{\mathbf{k}}$$

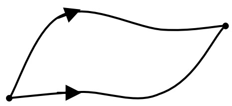
$$= \mathbf{0}$$

Thus, If $\mathbf{F} = \nabla \phi$, then $\nabla \times \mathbf{F} = \vec{0}$

Ex: Show that if $(\nabla \times \mathbf{F}) = 0 \Rightarrow$ there exists $\phi(x, y, z)$, s.t. $\mathbf{F} = \nabla \phi$

i..e., how do we find $\phi(x, y, z)$?

Proof: $\because \nabla \times \mathbf{F} = 0 \Rightarrow \int_C \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$ is path independent (by Stokes' theroem)



B (x, y, z) then $\int_A^B \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = f(B, A) = \bar{f}(x, y, z, x_0, y_0, z_0)$

A (x_0, y_0, z_0)

$$\Rightarrow \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \psi(x, y, z) - \psi(x_0, y_0, z_0)$$

Thus we can choose $\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$

See textbook (page 116~118) for more detailed proof of $\mathbf{F} = \bar{\nabla} \phi$

- Similarly if $\nabla \times \mathbf{F} = 0 \Rightarrow$ there exists $\phi(x, y, z)$ s.t. $\nabla \phi = \mathbf{F}$

i.e., $\begin{cases} \frac{\partial \phi}{\partial x} = F_x \\ \frac{\partial \phi}{\partial y} = F_y \\ \frac{\partial \phi}{\partial z} = F_z \end{cases}$, there exists a scalar function $\phi(x, y, z)$

Ex: For $\mathbf{F}=\dfrac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{x^2+y^2+z^2}$, then $\nabla\times\mathbf{F}=0$

Find out the corresponding $\phi(x,y,z)$ for $\nabla\phi=\mathbf{F}$

Sol: $$\frac{\partial\phi}{\partial x}=F_x=\frac{x}{x^2+y^2+z^2}\Rightarrow\phi=\int\left(\frac{x}{x^2+y^2+z^2}\right)dx+g(y,z)$$
$$=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)+g(y,z)\qquad(1)$$

$$\frac{\partial\phi}{\partial y}=F_y=\frac{y}{x^2+y^2+z^2}\qquad(2)$$

$$\text{substituting (1) into (2)}\Rightarrow\frac{y}{x^2+y^2+z^2}=\frac{y}{x^2+y^2+z^2}+\frac{\partial g}{\partial y}$$

$$\Rightarrow\frac{\partial g}{\partial y}=0\ \text{ or }\ g(y,z)=f(z)\ \text{ only}\qquad(3)$$

$$\frac{\partial\phi}{\partial z}=F_z=\frac{z}{x^2+y^2+z^2}\qquad(4)$$

$$\text{Substituting (1),(3) into (4)}\Rightarrow\frac{z}{x^2+y^2+z^2}=\frac{z}{x^2+y^2+z^2}+\frac{\partial f}{\partial z}\quad\therefore f=\text{const.}\ (5)$$

$$\text{Substituting (3),(5) into (1)}\Rightarrow\phi=\frac{1}{2}\ln\left(x^2+y^2+z^2\right)+c$$

Ex: The Maxwell's equations are

$$\nabla\cdot\mathbf{E}=4\pi\rho,\quad\nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\ \bar{\nabla}\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{j}\qquad(4.11)$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, ρ the charge density, $\mathbf{j}$ the current density, and c thr speed of light

We can use the Maxwell's equations to derive the charge continuity equation:

$$\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}=0\qquad(4.12)$$

Note that if there is no changing magnetic field in the Maxwell's equations

$$\begin{cases}\nabla\times\mathbf{E}=0 & \text{(A)}\quad(\because\mathbf{B}\text{ is not changing})\\ \nabla\cdot\mathbf{E}=4\pi\rho & \text{(B)}\end{cases}$$

From (B) $\Rightarrow \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = 4\pi\rho \Rightarrow$ three components of $\mathbf{E}(E_x, E_y, E_z)$

$\Rightarrow$ hard to find $\mathbf{E}$ exclusively !

From (A) $\Rightarrow$ we can guess $\mathbf{E} = \nabla\psi$, where $\psi = \psi(x, y, z)$

$\Rightarrow$ let us assume $\mathbf{E} = -\nabla\phi$, where $\phi = \phi(x, y, z) =$ electrical potential (C)

Substituting (C) into (B) $\Rightarrow \nabla \cdot (-\nabla\phi) = 4\pi\rho$

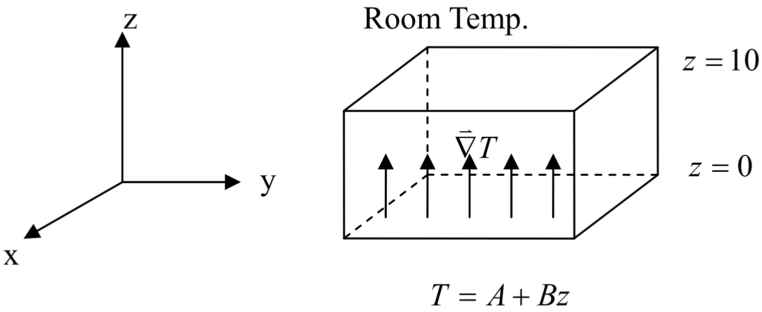
$$\Rightarrow -\nabla^2\phi = 4\pi\rho$$

$$\Rightarrow \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -4\pi\rho \qquad \text{(D)}$$

Note that Eq.(D) is one PDE for a single dependent variable ϕ!

Note: "$\nabla^2\Phi = 0$" is called the Laplace equation
"$\nabla^2\Phi = f(x, y, z)$" is called the Poisson equation

Example:



Temperature $= T(x, y, z)$

$T = A + Bz$

$\Rightarrow \nabla T = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + B\hat{\mathbf{k}}$

direction of $\vec{\nabla}T$: direction of maximum increase of T
magnitude of $\vec{\nabla}T$: rate of charge of T in space

- For incompressible fluid,

$\mathbf{V}(x, y, z) =$ velocity field

$\rho =$ constand ($\because$ incompressible flow)

From the continuity equation, $\dfrac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) = 0 \Rightarrow \nabla \cdot \mathbf{V} = 0$

$\because \rho =$ const.

If the flow is also irrotational, that is,

$$\nabla \times \mathbf{V} = 0 \Rightarrow \mathbf{V} = \nabla\phi$$

& incompressible flow $\nabla \cdot \mathbf{V} = 0 = \nabla \cdot (\nabla\phi) = \nabla^2\phi$

$\Rightarrow \nabla^2 \phi = 0$ Laplace equation for the velocity potential function $\phi(x, y, z)$

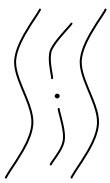
- Recall the Leibnitz Law:

$$\frac{\partial}{\partial x}\int_{u(x)}^{v(x)} f(t)\,dt = f(v)\frac{\partial v}{\partial x} - f(u)\frac{\partial u}{\partial x} \qquad (4.13)$$

For its limiting case: $[v(x) = u(x) = x]$

$$\int_{x_0}^{x} \frac{df(t)}{dt}\,dt = f(x) - f(x_0)$$

- For a stationary fluid



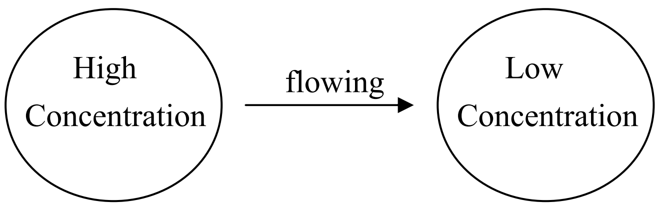
$\rho(x, y, z, t) =$ dye concentration

Then the fluid flows according to the Fick's Law

$$\mathbf{j} = -k\nabla\rho \qquad (4.14)$$

Note that from the continuity equation, mass flow $\mathbf{j} = \rho\mathbf{V}$

By diffusion, the fluid will move from high concentration to low concentration



$\because$ mass of dye is conserved

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0$$

$\Downarrow$ the Fick's Law, $\vec{\tau} = \rho\mathbf{V} = -k\nabla\rho$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(-k\nabla\rho) = 0$$

$$\therefore \quad \frac{\partial\rho}{\partial t} = k\nabla^2\rho \qquad \text{Diffusion equation} \qquad (4.15)$$

- Show that a diffusive process is irrotational,

  i.e., to prove $\nabla \times \mathbf{V} = 0$ for $\boldsymbol{\tau} = -k\nabla\rho = \rho\mathbf{V}$

Proof :

$$\mathbf{V} = -\frac{k}{\rho}\nabla\rho$$

$$\nabla \times \mathbf{V} = -\nabla \times \left(\frac{k}{\rho}\nabla\rho\right)$$

$$\frac{k}{\rho}\nabla\rho = k\left[\frac{1}{\rho}\frac{\partial\rho}{\partial x}\hat{\mathbf{i}} + \frac{1}{\rho}\frac{\partial\rho}{\partial y}\hat{\mathbf{j}} + \frac{1}{\rho}\frac{\partial\rho}{\partial z}\hat{\mathbf{k}}\right]$$

$$= k\left[\frac{\partial}{\partial x}(\ln\rho)\hat{\mathbf{i}} + \frac{\partial}{\partial y}(\ln\rho)\hat{\mathbf{j}} + \frac{\partial}{\partial z}(\ln\rho)\hat{\mathbf{k}}\right]$$

$$= k\nabla\ln\rho$$

$$\nabla \times \mathbf{V} = -k\nabla \times (\nabla\ln\rho) = -k\,\text{curl}\,(\text{grad}\ln\rho) = 0$$

- Directional Derivative:

Consider $f(x,y,z)$, a scalar field,
How does the value of $f(x,y,z)$ change or move in a particular direction?

Approach:

To find a path which is defined as

$$\text{PATH} \equiv \begin{cases} x(s) \\ y(s), \quad s = \text{arc length} \\ z(s) \end{cases}$$

$$\frac{df}{ds} = \frac{d}{ds}[f(x(s), y(s), z(s))]$$

$$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\Rightarrow \frac{df}{ds} = \underbrace{(\nabla f)}_{\text{gradient } f}\cdot\underbrace{\left(\frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{j}} + \frac{dz}{ds}\hat{\mathbf{k}}\right)}_{\text{the unit tangent vector to the curve} \equiv \hat{\mathbf{u}}} \qquad (4.16)$$

Hence,

$$\frac{df}{ds} = \nabla f \cdot \hat{\mathbf{u}} = \text{directional derivative of } f \text{ (at } \hat{\mathbf{u}} \text{ direction)}$$

$$= \text{rate of change of } f \text{ in the direction of } \hat{\mathbf{u}}$$

If we want to maximize $\frac{df}{ds} \Rightarrow$

$$\because \frac{df}{ds} = \nabla f \cdot \hat{\mathbf{u}} = |\nabla f||\hat{\mathbf{u}}|\cos\theta$$

To find the maximum of $\frac{df}{ds} \Rightarrow \nabla f \cdot \hat{\mathbf{u}}$ is maximized at $\theta = 0°$ $(\cos 0° = 1)$

$\Rightarrow \nabla f$ = direction of maximum rate of change of f

- Consider "$f(x, y, z) = \text{constant}$" → it is a "surface"

e.g. a constant temperature surface like "$T = 24$℃"

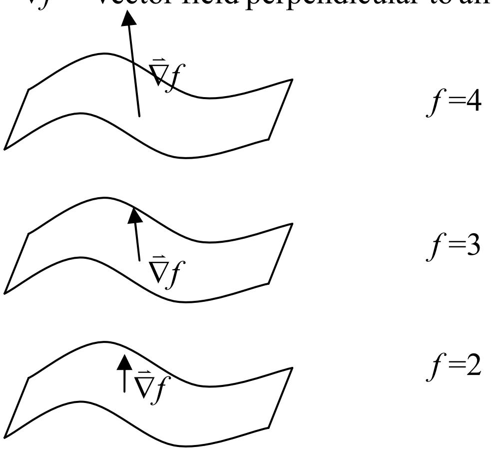
Take some curve C along this surface

then $\frac{df}{ds} = 0$ on the curve

$\Rightarrow \nabla f \cdot \hat{\mathbf{u}} = 0$ where $\hat{\mathbf{u}}$ is the tangent vector to surface "$f(x, y, z) = \text{constant}$"

$\Rightarrow \nabla f \perp \hat{\mathbf{u}}$

i.e., ∇f = vector field perpendicular to all "$f(x, y, z) = \text{constant}$" sfc.



- Integration by parts:

1D: $$\int_a^b u\,dv = \int_a^b d(uv) - \int_a^b v\,du = [uv]_a^b - \int_a^b v\,du$$

3D: $$\iiint_V \phi[\nabla \cdot \mathbf{F}]\,dV = \iiint_V [\nabla \cdot (\phi\,\mathbf{F}) - \nabla\phi \cdot \mathbf{F}]\,dV$$

(Integration by parts in multiple dimensions)

Using the divergence theorem, the above equation can be written as

$$\iiint_V \phi[\nabla\cdot\mathbf{F}]\,dV = \iiint_V [\nabla\cdot(\phi\,\mathbf{F}) - \nabla\phi\cdot\mathbf{F}]\,dV$$

$$= \iint_S \phi\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dA - \iiint_V \nabla\phi\cdot\mathbf{F}\,dV \qquad (4.17)$$

- If we choose $\mathbf{F} = \nabla\psi$

  RHS of (4.17): $\iint_S \phi\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \iint_S \phi\nabla\psi\cdot\hat{\mathbf{n}}\,dA$

  So Eq. (4.17) can be re-written as

$$\underbrace{\iiint_V \phi[\nabla\cdot\mathbf{F}]\,dV + \iiint_V \nabla\phi\cdot\mathbf{F}\,dV}\; = \underbrace{\iint_S \phi\,\mathbf{F}\cdot\hat{\mathbf{n}}\,dA} \qquad (4.18)$$

$$= \iiint_V \phi[\nabla\cdot\nabla\psi]\,dV + \iiint_V \nabla\phi\cdot\nabla\psi\,dV \qquad = \iint_S \phi\nabla\psi\cdot\hat{\mathbf{n}}$$

$$= \iiint_V \phi\nabla^2\psi\,dV + \iiint_V \nabla\phi\cdot\nabla\psi\,dV$$

So Eq.(4.18) can be written as

$$\iint_S \phi\nabla\psi\cdot\hat{\mathbf{n}}\,dA = \iiint_V \left[\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right]dV \qquad (4.19)$$

Green's First Identity
(First Form of Green Theorem)

- We can rewrite Eq. (4.19) by switching ϕ and ψ :

$$\iint_S \psi\nabla\phi\cdot\hat{\mathbf{n}}\,dA = \iiint_V \left[\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi\right]dV \qquad (4.20)$$

Eq.(4.19)–Eq.(4.20):

$$\iint_S \left[\phi\nabla\psi - \psi\nabla\phi\right]\cdot\hat{\mathbf{n}}\,dA = \iiint_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]dV \qquad (4.21)$$

Green's Second Identity
(Second Form of Green Theorem)

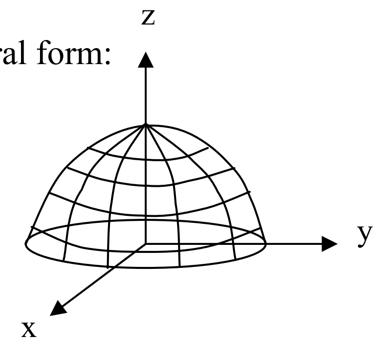
- The mathematical expression of a surface:

$$z = \bar{f}(x,y) \quad \leftrightarrow \quad \begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

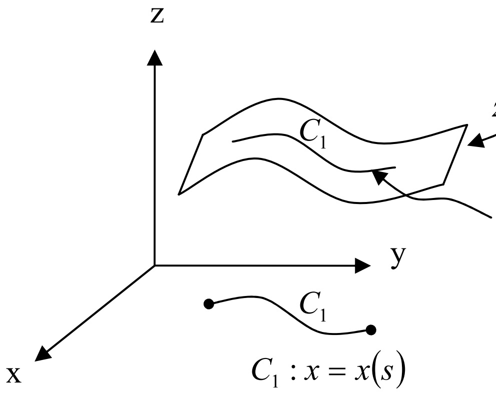
Particular form:

$$\begin{cases} x = u \\ y = v \\ z = \bar{f}(u,v) \end{cases}$$

general form:



- Directional Derivative:



A scalar function $\psi(x,y,z)$

$z = f(x,y)$

$C_1 : x = x(s)$
$\quad\; y = y(s)$

$$z = f(x,y) = f(x(s),y(s)) = g(s)$$

$$\Rightarrow \frac{d\psi}{ds} = \nabla\psi \cdot \left(\frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \right)$$

Something about Partial Differential Equation (PDE):

For a P.D.E. $\nabla^2\phi = 0$ or $\nabla^2\phi = f(x,y,z)$

(Laplace Eq.) (Poisson Eq.)

Boundary conditions:

if $\phi = g(x,y,z)$ on the boundary $\Leftarrow$ Dirichlet problem

if $\dfrac{\partial\phi}{\partial n} = g(x,y,z)$ on the boundary $\Leftarrow$ Neumann problem

Then one can solve the PDE by using the Green's function (theorem)
i.e., choosing appropriate $\psi(x,y,z)$ (eigen function)