Chapter 3 Surface Integral

Assume the surface is defined by

$$\left.\begin{aligned} x &= f(u,v) \\ y &= g(u,v) \\ z &= h(u,v) \end{aligned}\right\}, \text{ or, } z = \bar{f}(x,y)$$

then let us find the surface integral $\iint_S G(x,y,z)\,dA = ?$

often *G(x,y,z)* can be written as $G(x,y,z) = \mathbf{F}(x,y,z)\cdot\hat{\mathbf{n}}$

where $\hat{\mathbf{n}}$ is the normal vector of surface S,
and $\mathbf{F}$ is a vector function

Example: the gravitational force field is defined as

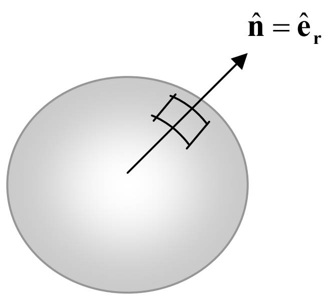
$$\mathbf{F}(r,\lambda,\phi) = -\frac{GMm}{r^2}\hat{\mathbf{e}}_\mathbf{r}$$

The fluid velocity is defined as $\mathbf{V}(x,y,z)$

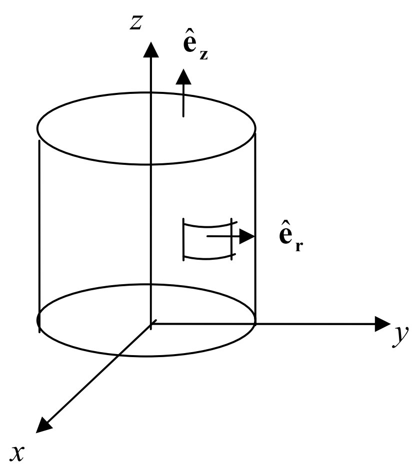
Note: $\hat{\mathbf{n}} \equiv$ unit normal vector to surface S
i.e., length = 1, and "$\perp$"(normal) to surface S

Ex:

Sphere



Cylinder



Find $\iint \mathbf{F}(x,y,z)\cdot \hat{\mathbf{n}}\, dA = ?$

$\Rightarrow$ Get two tangent vector $\mathbf{b}_{\mathrm{u}}, \mathbf{b}_{\mathrm{v}}$

$$\begin{cases} \mathbf{b}_{\mathrm{u}} = \dfrac{\partial f}{\partial u}\hat{\mathbf{i}} + \dfrac{\partial g}{\partial u}\hat{\mathbf{j}} + \dfrac{\partial h}{\partial u}\hat{\mathbf{k}} \\ \mathbf{b}_{\mathrm{v}} = \dfrac{\partial f}{\partial v}\hat{\mathbf{i}} + \dfrac{\partial g}{\partial v}\hat{\mathbf{j}} + \dfrac{\partial h}{\partial v}\hat{\mathbf{k}} \end{cases}$$

then by using $\mathbf{b}_{\mathrm{u}}$ and $\mathbf{b}_{\mathrm{v}}$, we can find normal vector $\hat{\mathbf{n}}$ as

$$\mathbf{N} = \mathbf{b}_{\mathrm{u}} \times \mathbf{b}_{\mathrm{v}}$$

$$\therefore \mathbf{n} = \pm \frac{\mathbf{b}_{\mathrm{u}} \times \mathbf{b}_{\mathrm{v}}}{\left|\mathbf{b}_{\mathrm{u}} \times \mathbf{b}_{\mathrm{v}}\right|}$$

("$\perp$"depends on the problem, but $\hat{\mathbf{n}}$ always directs outward from the surface)

$$dA = \left|\mathbf{b}_{\mathrm{u}} \times \mathbf{b}_{\mathrm{v}}\right| du dv$$

$$\Rightarrow \iint_S \mathbf{F}(x,y,z)\cdot \hat{\mathbf{n}}\, dA$$

$$= \iint_S \mathbf{F}(f(u,v), g(u,v), h(u,v))\cdot(\mathbf{b}_{\mathrm{u}} \times \mathbf{b}_{\mathrm{v}})du dv \qquad (3.1)$$

need to compute each component

Ex: For $z = \bar{f}(x,y)$,

$$\Rightarrow \begin{cases} x = x \\ y = y \\ z = \bar{f}(x,y) \end{cases} \qquad \begin{aligned} \mathbf{b}_{\mathrm{x}} &= \hat{\mathbf{i}} + \frac{\partial \bar{f}}{\partial x}\hat{\mathbf{k}} \\ \mathbf{b}_{\mathrm{y}} &= \hat{\mathbf{j}} + \frac{\partial \bar{f}}{\partial y}\hat{\mathbf{k}} \end{aligned}$$

$$\Rightarrow \mathbf{b}_{\mathrm{x}} \times \mathbf{b}_{\mathrm{y}} = \mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \dfrac{\partial \bar{f}}{\partial x} \\ 0 & 1 & \dfrac{\partial \bar{f}}{\partial y} \end{vmatrix} = -\frac{\partial \bar{f}}{\partial x}\hat{\mathbf{i}} - \frac{\partial \bar{f}}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\Rightarrow \hat{\mathbf{n}} = \frac{\mathbf{N}}{\left[\left(\frac{\partial \bar{f}}{\partial x}\right)^2 + \left(\frac{\partial \bar{f}}{\partial y}\right)^2 + 1\right]^{1/2}} = \frac{-\frac{\partial \bar{f}}{\partial x}\hat{\mathbf{i}} - \frac{\partial \bar{f}}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}}}{\left[\left(\frac{\partial \bar{f}}{\partial x}\right)^2 + \left(\frac{\partial \bar{f}}{\partial y}\right)^2 + 1\right]^{1/2}}$$

$$\therefore \iint_S \mathbf{F}(x,y,z) \cdot \hat{\mathbf{n}}\, dA = \iint_{S(x,y)} \mathbf{F}\left(x,y,\bar{f}(x,y)\right) \cdot \left(-\frac{\partial \bar{f}}{\partial x}, -\frac{\partial \bar{f}}{\partial y}, 1\right) dxdy$$

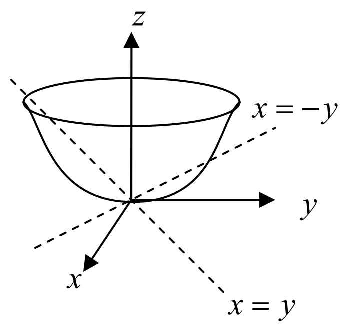
Note: the physical meaning of $\mathbf{F} \cdot \hat{\mathbf{n}} \equiv$ the flux of $\mathbf{F}$ through surface S

Ex: Consider a vector field $\mathbf{F} = \hat{\mathbf{i}} + xy\,\hat{\mathbf{j}}$

a surface S: $\left.\begin{aligned} x &= u + v \\ y &= u - v \\ z &= u^2 \end{aligned}\right\} \begin{aligned} 0 \le u \le 1 \\ 0 \le v \le 1 \end{aligned}$

Find the surface integral $\iint_S \mathbf{F} \cdot \vec{\mathbf{n}}\, dA = ?$

Sol: $\because \frac{x+y}{2} = u \qquad \therefore z = \frac{(x+y)^2}{4}$



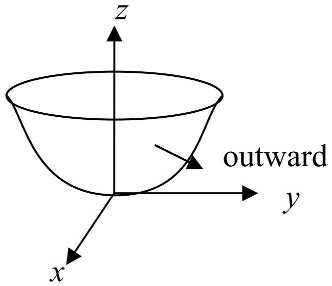
$$\mathbf{b}_{\mathbf{u}} = \frac{\partial f}{\partial u}\hat{\mathbf{i}} + \frac{\partial g}{\partial u}\hat{\mathbf{j}} + \frac{\partial h}{\partial u}\hat{\mathbf{k}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 2u\hat{\mathbf{k}}$$

$$\mathbf{b}_{\mathbf{v}} = \frac{\partial f}{\partial v}\hat{\mathbf{i}} + \frac{\partial g}{\partial v}\hat{\mathbf{j}} + \frac{\partial h}{\partial v}\hat{\mathbf{k}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

$$\begin{aligned}
\therefore \iint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA &= \iint_{S(u,v)} \mathbf{F} \cdot (\mathbf{b}_{\mathbf{u}} \times \mathbf{b}_{\mathbf{v}}) dudv \\
&= \iint \left[\hat{\mathbf{i}} + (u^2 - v^2)\hat{\mathbf{j}}\right] \cdot [\mathbf{b}_{\mathbf{u}} \times \mathbf{b}_{\mathbf{v}}] dudv \\
&= \int_0^1 \int_0^1 [1, u^2 - v^2, 0] \cdot [2u, 2u, -2]\, dudv \\
&= \int_0^1 \int_0^1 \left[2u + 2u(u^2 - v^2)\right] dudv = \int_0^1 \int_0^1 \left[2u + 2u^3 - 2uv^2\right] dudv \\
&= \int_0^1 \left[u^2 + \frac{1}{2}u^4 - u^2 v^2\right]_{u=0}^{u=1} dv = \int_0^1 \left[1 + \frac{1}{2} - v^2\right] dv = \frac{3}{2} - \frac{1}{3} = \frac{7}{6}
\end{aligned}$$

$$\left(\mathbf{b_u} \times \mathbf{b_v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 2u \\ 1 & -1 & 0 \end{vmatrix} = +2u\hat{\mathbf{i}} + 2u\hat{\mathbf{j}} + (-2u)\hat{\mathbf{k}} \right)$$

Ex:



$$x^2 + y^2 \le 1, \quad z = \frac{x^2 + y^2}{4}$$

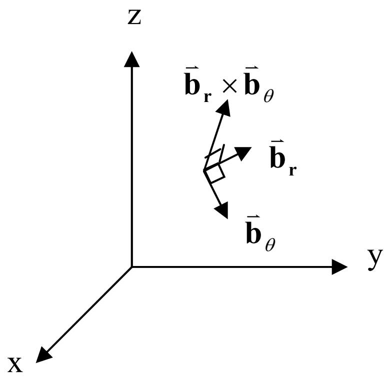
For $\mathbf{F} = \hat{\mathbf{i}} + xy\hat{\mathbf{j}}$

Find $\iint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = ?$

Sol: Transform the Cartesian coordinate into the cylindrical coordinate

$$\begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{matrix} \Rightarrow \begin{cases} x = r\cos\theta = f(r,\theta) \\ y = r\sin\theta = g(r,\theta) \\ z = \dfrac{x^2 + y^2}{4} = \dfrac{r^2}{4} = h(r,\theta) \end{cases}$$

$$\Rightarrow \begin{matrix} \mathbf{b_r} = \dfrac{\partial f}{\partial r}\hat{\mathbf{i}} + \dfrac{\partial g}{\partial r}\hat{\mathbf{j}} + \dfrac{\partial h}{\partial r}\hat{\mathbf{k}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} + \dfrac{r}{2}\hat{\mathbf{k}} \\ \mathbf{b}_\theta = \dfrac{\partial f}{\partial \theta}\hat{\mathbf{i}} + \dfrac{\partial g}{\partial \theta}\hat{\mathbf{j}} + \dfrac{\partial h}{\partial \theta}\hat{\mathbf{k}} = -r\sin\theta\,\hat{\mathbf{i}} + r\cos\theta\,\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \end{matrix}$$



Thus, Find $\iint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \iint \mathbf{F} \cdot (\mathbf{b}_\theta \times \mathbf{b_r})\, dr d\theta$

Note that

$$\mathbf{F} \cdot (\mathbf{b}_\theta \times \mathbf{b_r}) = \begin{vmatrix} 1 & r^2\cos\theta\sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & \dfrac{r}{2} \end{vmatrix}$$

$$= \frac{r^2}{2}\cos\theta - r^2\cos\theta\sin\theta\left(-\frac{r^2}{2}\sin\theta\right)$$

$$= \frac{r^2}{2}\cos\theta + \frac{r^4}{2}\cos\theta\sin^2\theta$$

$$\Rightarrow \iint \mathbf{F}\cdot\hat{\mathbf{n}}\,dA=\int_0^{2\pi}\int_0^1\left[\frac{r^2}{2}\cos\theta+\frac{r^4}{2}\cos\theta\sin^2\theta\right]dr\,d\theta$$
$$=\int_0^{2\pi}\left[\frac{1}{6}\cos\theta+\frac{1}{10}\cos\theta\sin^2\theta\right]d\theta$$
$$=\frac{1}{6}\int_0^{2\pi}\cos\theta\,d\theta+\frac{1}{10}\int_0^{2\pi}\sin^2\theta\,d\sin\theta$$
$$=\frac{1}{6}\cdot 0+\frac{1}{30}(0-0)=0$$

● Divergence of Vector fields

Recall the Gauss's Law of Electrostatics:

$$\iint_S \mathbf{E}\cdot\vec{\mathbf{n}}dA=4\pi q_o \qquad (3.2)$$

where S is any closed surface,

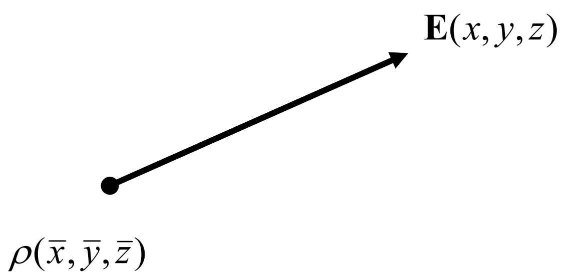
$\mathbf{E}$ is the electrical field,

$\hat{\mathbf{n}}$ is the normal vector of surface S,

q_o is the total electrical charge contained by surface S.

Another way to find the electrical field $\mathbf{E}$:

$$\mathbf{E}(x,y,z)=\iiint\frac{\rho(\bar{x},\bar{y},\bar{z})\hat{u}(r-\bar{r})}{(x-\bar{x})^2+(y-\bar{y})^2+(z-\bar{z})^2}d\bar{x}d\bar{y}d\bar{z}$$



where $\rho(\bar{x},\bar{y},\bar{z})$ is the charge density

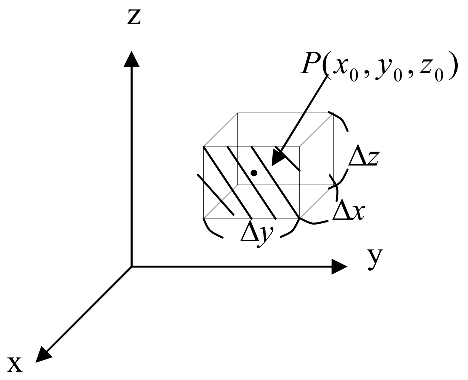
Consider $\mathbf{E}$ changes smoothly and ρ also changes smoothly

Thus $\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}dA=4\pi q_o\approx 4\pi\rho(x,y,z)\Delta V$

$$\Rightarrow \lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S}\mathbf{E}\cdot\hat{\mathbf{n}}dA=4\pi\rho(x,y,z)=div\mathbf{E}=\nabla\cdot\mathbf{E}$$

Thus

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{E} \cdot \hat{\mathbf{n}} dA = \nabla \cdot \mathbf{E} \qquad (3.3)$$



$\Delta V = \Delta x \Delta y \Delta z$, $P(x_0, y_0, z_0)$: center of cubic

Consider $\mathbf{E} = E_x \hat{\mathbf{i}} + E_y \hat{\mathbf{j}} + E_z \hat{\mathbf{k}}$

for the front surface $\Rightarrow$ $\hat{\mathbf{n}} = \hat{\mathbf{i}}$

for the back surface $\Rightarrow$ $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$

Than at the front surface,

$$\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{E}\left(x_0 + \frac{\Delta x}{2}, y_0, z_0 \right) \cdot \left(\hat{\mathbf{i}}\right) = E_x\left(x_0 + \frac{\Delta x}{2} \right)$$

$$\Delta A = \Delta y \Delta z$$

$$\therefore \iint_{FRONT} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = E_x\left(x_0 + \frac{\Delta x}{2} \right) \Delta y \Delta z$$

Similarly, $\displaystyle \iint_{\text{BACK}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = -E_x\left(x_0 - \frac{\Delta x}{2} \right) \Delta y \Delta z$

$$\Rightarrow \iint_{FRONT} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA + \iint_{\text{BACK}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = \left[E_x\left(x_0 + \frac{\Delta x}{2} \right) - E_x\left(x_0 - \frac{\Delta x}{2} \right) \right] \Delta y \Delta z \qquad (A)$$

$$\frac{(A)}{\Delta V} \Rightarrow \frac{1}{\Delta V} \left\{ \iint_{FRONT} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA + \iint_{\text{BACK}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA \right\} = \frac{1}{\Delta x} \left[E_x\left(x_0 + \frac{\Delta x}{2} \right) - E_x\left(x_0 - \frac{\Delta x}{2} \right) \right]$$

$$(\Delta V = \Delta x \Delta y \Delta z) \quad \begin{pmatrix} \text{further assume} \\ \Delta V \to 0 \text{ i.e.}, \Delta x \to 0, \\ \Delta y \to 0, \Delta z \to 0 \end{pmatrix} \qquad = \frac{\partial E_x}{\partial x}(x_0, y_0, z_0)$$

Similarly for other two surface (LEFT & RIGHT, UP & DOWN)

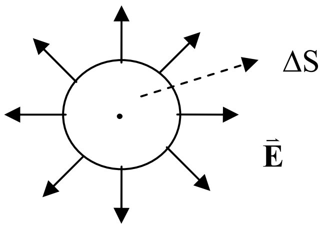
$$\Rightarrow \nabla \cdot \mathbf{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_{\Delta S} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.4)$$

where $\mathbf{E} = E_x \hat{\mathbf{i}} + E_y \hat{\mathbf{j}} + E_z \hat{\mathbf{k}}$

∴ Gauss's Law can be written as

$$\boxed{\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=4\pi\rho(x,y,z)} \qquad (3.5)$$

Physical meaning of $\nabla\cdot\mathbf{E}$:



$$\nabla\cdot\mathbf{E}=\lim_{\Delta V\to 0}\frac{1}{\Delta V}\iint_{\Delta S}\mathbf{E}\cdot\hat{\mathbf{n}}\,dA$$

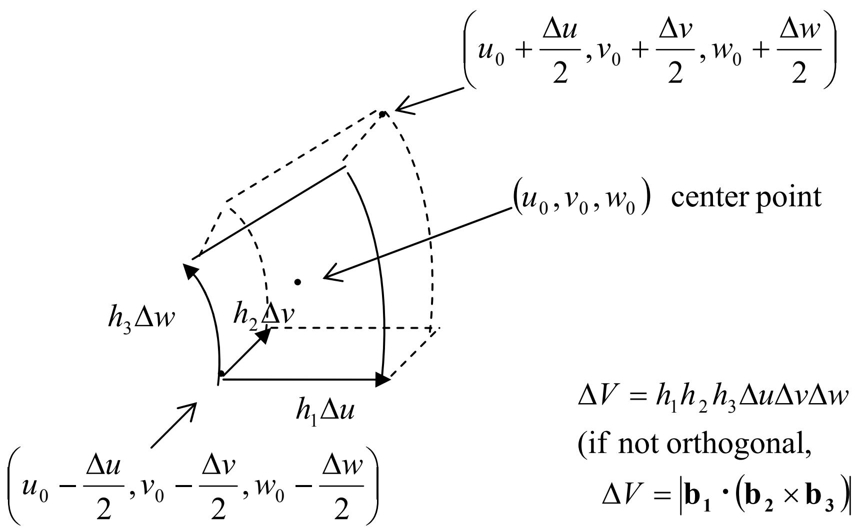
⇧ Divergence of vector field $\vec{\mathbf{E}}$ (per unit volume)

Divergence of $\mathbf{F}$ in Cartesian coordinate:

$$\nabla\cdot\mathbf{F}=\frac{\partial F_x}{\partial x}+\frac{\partial F_{yx}}{\partial y}+\frac{\partial F_z}{\partial z},\quad \text{where } \mathbf{F}=F_x\hat{\mathbf{i}}+F_y\hat{\mathbf{j}}+F_z\hat{\mathbf{k}}$$

For a general orthogonal coordinate system (u,v,w)

$$\begin{cases} x=f(u,v,w)\\ y=g(u,v,w)\\ z=h(u,v,w)\end{cases} \qquad \mathbf{b_1},\mathbf{b_2},\mathbf{b_3} \text{ with length } h_1,h_2,h_3$$



$\Delta V=h_1h_2h_3\Delta u\Delta v\Delta w$ (∵ orthogonal)

(if not orthogonal,

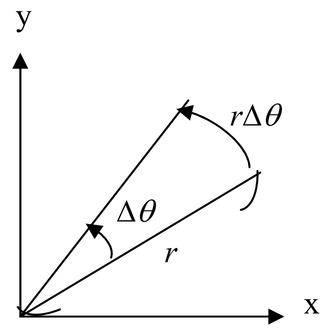
$\Delta V=\left|\mathbf{b_1}\cdot(\mathbf{b_2}\times\mathbf{b_3})\right|dudvdw$

By definition,

$$\nabla \cdot \mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA$$

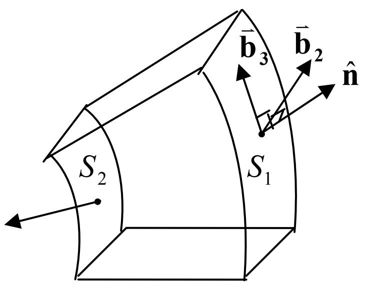
$$= \lim_{\Delta V \to 0} \frac{1}{h_1 h_2 h_3 \Delta u \Delta v \Delta w} \iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA$$

Ex: Consider the polar coordinate (r, θ)



$$h_1 \Delta u = \Delta r \qquad (h_1 = 1)$$

$$h_2 \Delta v = r\Delta\theta \qquad (h_2 = \mathrm{r})$$



Consider these two surfaces

$$S_1 : "u = u_0 + \frac{\Delta u}{2}" \text{constant surface}$$

$$S_2 : "u = u_0 - \frac{\Delta u}{2}" \text{constant surface}$$

$$\hat{\mathbf{n}} = \frac{\mathbf{b_2} \times \mathbf{b_3}}{|\mathbf{b_2} \times \mathbf{b_3}|}, \quad \Delta A = |\mathbf{b_2} \times \mathbf{b_3}| \Delta v \Delta w$$

Then $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = F\left(u_0 + \frac{\Delta u}{2}, v_0, w_0\right) \cdot \frac{\mathbf{b_2} \times \mathbf{b_3}}{|\mathbf{b_2} \times \mathbf{b_3}|} |\mathbf{b_2} \times \mathbf{b_3}| \Delta v \Delta w$

$$= F\left(u_0 + \frac{\Delta u}{2}, v_0, w_0\right) \cdot (\mathbf{b_2} \times \mathbf{b_3}) \Delta v \Delta w$$

Decompose $\mathbf{F}$ into three components:

$$\mathbf{F} = F_u \hat{\mathbf{e}}_\mathbf{u} + F_v \hat{\mathbf{e}}_\mathbf{v} + F_w \hat{\mathbf{e}}_\mathbf{w}$$

Along surface S_1 :

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_\mathbf{u}, \ \mathbf{b_2} \times \mathbf{b_3} = h_2 h_3 \hat{\mathbf{e}}_\mathbf{u}$$

$$\therefore \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\, dA = F_u\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)h_2\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)h_3\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)\Delta v\Delta w$$

Similarly $\iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\, dA = -F_u\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)h_2\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)h_3\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)\Delta v\Delta w$

Thus, $\lim_{\Delta V\to 0}\frac{1}{\Delta V}\left\{\iint_{S_1}\mathbf{F}\cdot\hat{\mathbf{n}}\, dA+\iint_{S_2}\mathbf{F}\cdot\hat{\mathbf{n}}\, dA\right\}$

$$\begin{aligned}&=\lim_{\Delta V\to 0}\frac{\Delta v\Delta w}{h_1h_2h_3\Delta u\Delta v\Delta w}\left\{F_u\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)h_2\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)h_3\left(u_0+\frac{\Delta u}{2}, v_0, w_0\right)\right.\\
&\qquad\left.-F_u\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)h_2\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)h_3\left(u_0-\frac{\Delta u}{2}, v_0, w_0\right)\right\}\\
&=\frac{1}{h_1h_2h_3}\frac{\partial}{\partial u}\left(F_uh_2h_3\right)\end{aligned}$$

Similar derivations are for other four surfaces (UP & DOWN, FRONT & BACK)

Thus the final expression is

$$\nabla\cdot\mathbf{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u}\left(F_uh_2h_3\right)+\frac{\partial}{\partial v}\left(F_vh_3h_1\right)+\frac{\partial}{\partial w}\left(F_wh_1h_2\right)\right] \qquad (3.6)$$

Ex: For the Cylindrical coordinate (r,θ,z):

$h_1=1 \Leftrightarrow r-$component

$h_2=r \Leftrightarrow \theta-$component

$h3=1 \Leftrightarrow z-$component

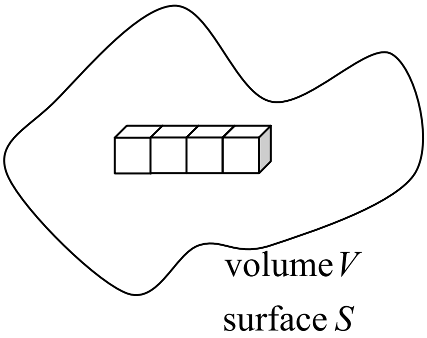
$$\begin{aligned}\therefore \nabla\cdot\mathbf{F}&=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(F_r\cdot r\right)+\frac{\partial}{\partial\theta}\left(F_\theta\right)+\frac{\partial}{\partial z}\left(F_z\cdot r\right)\right]\\
&=\frac{1}{r}\left[r\frac{\partial F_r}{\partial r}+F_r\right]+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}\\
&=\frac{\partial F_r}{\partial r}+\frac{1}{r}F_r+\frac{1}{r}\frac{\partial F_\theta}{\partial\theta}+\frac{\partial F_z}{\partial z}\end{aligned}$$

Divergence Theorem (or The Guess Theorem):

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \iiint_V \nabla \cdot \mathbf{F}\, dV \qquad (3.7)$$

where S is a surface

V is the volume contained by the closed surface S



$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \sum_{\substack{small\\volume}} \iint_{S_i} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA$$

$$= \lim_{\Delta V_i \to 0} \sum_{\substack{small\\volume}} \Delta V_i \left(\frac{1}{\Delta V_i} \iint \mathbf{F} \cdot \hat{\mathbf{n}}\, dA \right)$$

$$= \lim_{\Delta V_i \to 0} \sum_{\substack{small\\volume}} \Delta V_i (\nabla \cdot \mathbf{F})$$

$$= \iiint_V \nabla \cdot \mathbf{F}\, dV$$

Application of divergence theorem $\Rightarrow$ mass conservation

Let
density $= \rho(x, y, z, t)$
velocity $= \vec{\mathbf{V}}(x, y, z, t)$
$M(t) =$ total mass in the fixed volume

then $\dfrac{dM(t)}{dt} = \dfrac{d}{dt} \iiint_V \rho(x, y, z, t) dV = \iiint_V \dfrac{\partial \rho}{\partial t} dV$

(fixed volume V)

In other words,

$$\frac{dM(t)}{dt} = -\iint_S \rho \mathbf{V} \cdot \hat{\mathbf{n}}\, dA \qquad \text{(flux outward of } \rho\mathbf{V}\text{)}$$

$$= -\iiint_V \nabla \cdot (\rho \mathbf{V}) dV \qquad \text{(using the divergence theorem)}$$

$$\therefore \iiint_V \frac{\partial \rho}{\partial t} dV = -\iiint_V \nabla \cdot (\rho \mathbf{V}) dV$$

Thus for any volume $V \Rightarrow \dfrac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{V}) \qquad (3.8) \qquad$ Continuity Equation

Note that for incompressible fluid, $\rho = \text{constant}, \text{i.e.,} \frac{\partial \rho}{\partial \text{t}} = 0$

So the continuity equation is reduced to

$$0 = \nabla \cdot (\rho \mathbf{V}) = \rho \nabla \cdot \mathbf{V} \Rightarrow \quad \nabla \cdot \mathbf{V} = 0 \qquad (3.9)$$

Continuity Equation for Incompressible fluid