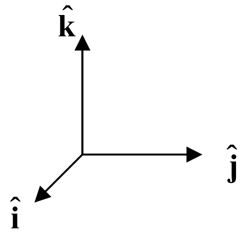
Chapter 2 Coordinate Systems

Definition: A coordinate system is simple a systematic method of labeling points in a set (such as $\mathbf{R}^2, \mathbf{R}^3$, points on a globe).

For a Euclidean space (e.g. $\mathbf{R}^2, \mathbf{R}^3$),
Cartesian coordinate system is the easiest coordinate.

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (2.1)$$



where $\mathbf{A} = (a_1, a_2, a_3)$, $\mathbf{B} = (b_1, b_2, b_3)$

and $\mathbf{A} \cdot \mathbf{B} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i \qquad (2.2)$

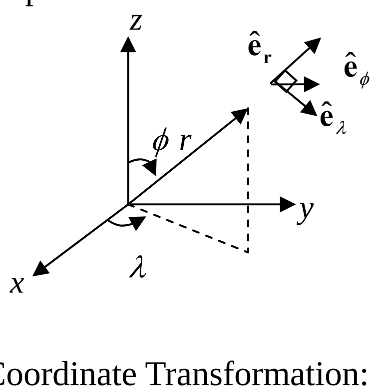
$$\frac{d\mathbf{A}}{dt} = \frac{da_1}{dt}\hat{\mathbf{i}} + \frac{da_2}{dt}\hat{\mathbf{j}} + \frac{da_3}{dt}\hat{\mathbf{k}}$$

Curvilinear Coordinate Systems:

Reasons to use this coordinate system —

1. certain symmetry in the problem (e.g., centrifugal force)
2. working on a curved surface (e.g. navigating along the earth)

Example: Spherical Polar Coordinate (r, λ, ϕ)



$P(r, \lambda, \phi)$

unit vector: $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\lambda, \hat{\mathbf{e}}_\phi$

Question: How to express
$$\hat{\mathbf{e}}_r = l(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) = ?$$
$$\hat{\mathbf{e}}_\lambda = m(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) = ?$$
$$\hat{\mathbf{e}}_\phi = n(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) = ?$$

General Coordinate Transformation:

2D	3D
$x = f(u,v)$ $y = g(u,v)$	$x = f(u,v,w)$ $y = g(u,v,w)$ $z = f(u,v,w)$

Ex1: Polar Coordinate: (r, θ)

$$\Rightarrow \quad \begin{aligned} x &= u\cos v \\ y &= u\sin v \end{aligned}$$

(i.e., $u = r, v = \theta$)

Ex2: Cylindrical Coordinate: (r,θ,z)

$$\Rightarrow \quad \begin{aligned} x &= u\cos v \\ y &= u\sin v \\ z &= w \end{aligned}$$

(i.e., $u = r, v = \theta, w = z$)

Basis Unit Vectors: $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$

$\hat{\mathbf{e}}_1$ = unit vector associated with the direction change in u
$\hat{\mathbf{e}}_2$ = unit vector associated with the direction change in v
$\hat{\mathbf{e}}_3$ = unit vector associated with the direction change in w

What are $\hat{\mathbf{e}}_u$ in terms of $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$?

$\Rightarrow$ hold v, w fixed

$$dx = \frac{\partial f}{\partial u}du \qquad dy = \frac{\partial g}{\partial u}du \qquad dz = \frac{\partial h}{\partial u}du$$

, so

$$\begin{aligned} d\mathbf{r} &= (dx, dy, dz) \\ &= \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}\right)du \\ &= \left(\frac{\partial f}{\partial u}\hat{\mathbf{i}} + \frac{\partial g}{\partial u}\hat{\mathbf{j}} + \frac{\partial h}{\partial u}\hat{\mathbf{k}}\right)du = \mathbf{b}_1 du \qquad (2.2.1) \end{aligned}$$

$\Rightarrow$ Similarly, hold u, w fixed

$$d\mathbf{r} = \left(\frac{\partial f}{\partial v}\hat{\mathbf{i}} + \frac{\partial g}{\partial v}\hat{\mathbf{j}} + \frac{\partial h}{\partial v}\hat{\mathbf{k}}\right)dv = \mathbf{b}_2 dv \qquad (2.2.2)$$

$$d\mathbf{r} = \left(\frac{\partial f}{\partial w}\hat{\mathbf{i}} + \frac{\partial g}{\partial w}\hat{\mathbf{j}} + \frac{\partial h}{\partial w}\hat{\mathbf{k}}\right)dw = \mathbf{b}_3 dw \qquad (2.2.3)$$

- Unit Basis Vector:

$$\hat{\mathbf{e}}_\mathrm{i} = \frac{\mathbf{b}_\mathrm{i}}{|\mathbf{b}_\mathrm{i}|} \quad , i = 1, 2, 3 \text{ (or } u, v, w)$$

Orthogonal Coordinate Systems:

$$\hat{\mathbf{e}}_\mathrm{i} \cdot \hat{\mathbf{e}}_\mathrm{j} = 0 \quad , \mathbf{i} \neq \mathbf{j}$$

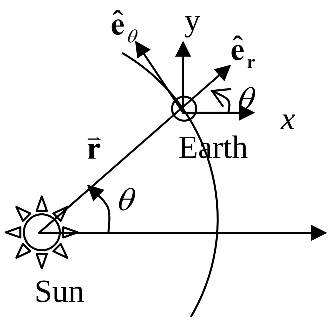
Example: Cylindrical Coordinates

$$\Rightarrow \begin{array}{l} x = u\cos v = f(u,v,w) \\ y = u\sin v = g(u,v,w) \\ z = \quad w \quad = h(u,v,w) \end{array}$$

$$\Rightarrow \begin{array}{lll} \mathbf{b}_1 = \cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}} & , |\mathbf{b}_1| = 1, & \hat{\mathbf{e}}_1 = \mathbf{b}_1 = \cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}} \\ \mathbf{b}_2 = -u\sin v\hat{\mathbf{i}} + u\cos v\hat{\mathbf{j}}, & |\mathbf{b}_2| = u, & \hat{\mathbf{e}}_2 = \mathbf{b}_2 / u = -\sin v\hat{\mathbf{i}} + \cos v\hat{\mathbf{j}} \\ \mathbf{b}_3 = 1\hat{\mathbf{k}} & , |\mathbf{b}_1| = 1, & \hat{\mathbf{e}}_3 = \mathbf{b}_3 = \hat{\mathbf{k}} \end{array}$$

Note that $\frac{d}{dt}\hat{\mathbf{i}} = 0$, but $\frac{d}{dt}\hat{\mathbf{e}}_\mathrm{i} \neq 0$ (since $\hat{\mathbf{e}}_\mathrm{i}$ could be rotating)

Ex:



Position of earth: $\mathrm{P} = (r(t), \theta(t))$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

And

$$\hat{\mathbf{e}}_r = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$

$$\hat{\mathbf{e}}_\lambda = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

Newton's Law of Motion: $\mathbf{F} = M_E\mathbf{a}$, where $\mathbf{F} = -\frac{GM_S M_E}{r^2}\hat{\mathbf{e}}_\mathbf{r}$

(M_S = Mass of Sun, M_E = Mass of Earth)

Position vector of earth: $\mathbf{r} = r\hat{\mathbf{e}}_\mathrm{r}$

Velocity vector of earth: $\mathbf{v} = \dfrac{d}{dt}\left(r\hat{\mathbf{e}}_\mathrm{r}\right) = \dot{r}\hat{\mathbf{e}}_\mathrm{r} + r\dot{\hat{\mathbf{e}}}_\mathrm{r}$

Note that

$$\dot{\hat{\mathbf{e}}}_\mathrm{r} = \frac{d}{dt}\left(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\right) = -\sin\theta\cdot\dot{\theta}\,\hat{\mathbf{i}} + \cos\theta\cdot\dot{\theta}\,\hat{\mathbf{j}} = \dot{\theta}\,\hat{\mathbf{e}}_\theta$$

$$\dot{\hat{\mathbf{e}}}_\theta = \frac{d}{dt}\left(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\right) = -\cos\theta\cdot\dot{\theta}\,\hat{\mathbf{i}} - \sin\theta\cdot\dot{\theta}\,\hat{\mathbf{j}} = -\dot{\theta}\,\hat{\mathbf{e}}_\mathrm{r}$$

$$\Rightarrow \quad \mathbf{v} = \dot{r}\hat{\mathbf{e}}_\mathrm{r} + r\dot{\theta}\,\hat{\mathbf{e}}_\theta$$

Similarly, acceleration vector of earth is

$$\begin{aligned}
\vec{\mathbf{a}} &= \frac{d}{dt}\vec{\mathbf{v}} = \frac{d}{dt}\left(\dot{r}\hat{\mathbf{e}}_\mathrm{r} + r\dot{\theta}\,\hat{\mathbf{e}}_\theta\right) \\
&= \ddot{r}\hat{\mathbf{e}}_\mathrm{r} + \dot{r}\underbrace{\dot{\hat{\mathbf{e}}}_\mathrm{r}}_{=\dot{\theta}\,\hat{\mathbf{e}}_\theta} + \left(\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\mathbf{e}}_\theta + r\dot{\theta}\underbrace{\dot{\hat{\mathbf{e}}}_\theta}_{=-\dot{\theta}\,\hat{\mathbf{e}}_\mathrm{r}} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{e}}_\mathrm{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\mathbf{e}}_\theta
\end{aligned}$$

centripetal acceleration

Equating $\mathbf{F} = M_E\mathbf{a}$, we obtain

$$\begin{cases} \ddot{r} - r\left(\dot{\theta}\right)^2 = -\dfrac{GM_S}{r^2} \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \end{cases}$$

We will solve this system of equation later

(see Chapter 6)

For General Curvilinear Coordinate:

$$\mathbf{A}(\underbrace{\mathbf{u}(\mathbf{t})}_{=(u,v,w)}, t) = A_1(\mathbf{u}(\mathbf{t}),t)\hat{\mathbf{e}}_1 + A_2(\mathbf{u}(\mathbf{t}),t)\hat{\mathbf{e}}_2 + A_3(\mathbf{u}(\mathbf{t}),t)\hat{\mathbf{e}}_3$$

Thus,

$$\frac{d\mathbf{A}}{dt} = \sum_{i=1}^{3}\left[\frac{\partial A_i}{\partial u}\dot{u} + \frac{\partial A_i}{\partial v}\dot{v} + \frac{\partial A_i}{\partial w}\dot{w} + \frac{\partial A_i}{\partial t}\right]\hat{\mathbf{e}}_\mathrm{i} + \sum_{i=1}^{3} A_i\left[\frac{\partial \hat{\mathbf{e}}_\mathrm{i}}{\partial u}\dot{u} + \frac{\partial \hat{\mathbf{e}}_\mathrm{i}}{\partial v}\dot{v} + \frac{\partial \hat{\mathbf{e}}_\mathrm{i}}{\partial w}\dot{w}\right] \qquad (2.3)$$

Find out $\dfrac{\partial \hat{\mathbf{e}}_\mathrm{i}}{\partial(u,v,w)} = \alpha\,\hat{\mathbf{e}}_1 + \beta\,\hat{\mathbf{e}}_2 + \gamma\,\hat{\mathbf{e}}_3$ (2.4)

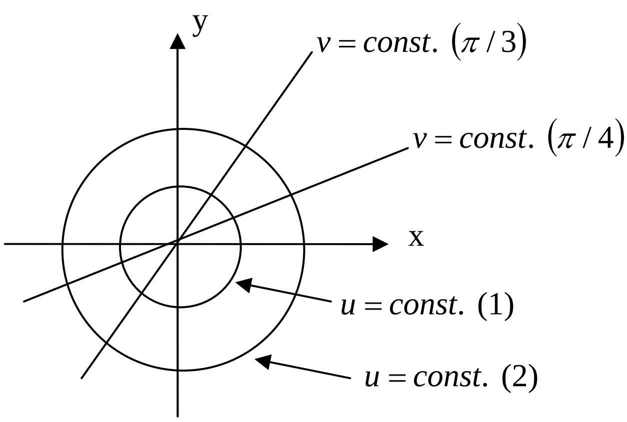
Substitute (2.4) into (2.3), we can get $\frac{d\mathbf{A}}{dt} = (\ \ )\hat{\mathbf{e}}_1 + (\ \ )\hat{\mathbf{e}}_2 + (\ \ )\hat{\mathbf{e}}_3$

Ex: For the polar coordinate (r,θ) or (u,v)

$$\begin{cases} x = u\cos v = f(u,v) \\ y = u\sin v = g(u,v) \end{cases}$$

To find $v = const. \Rightarrow \frac{x}{y} = \cot v$

To find $u = const. \Rightarrow x^2 + y^2 = u^2$



To find the basis vector:

$$\mathbf{b}_1 = \frac{\partial f}{\partial u}\hat{\mathbf{i}} + \frac{\partial g}{\partial u}\hat{\mathbf{j}} = \cos v\hat{\mathbf{i}} + \sin v\hat{\mathbf{j}}$$

$$\mathbf{b}_2 = \frac{\partial f}{\partial v}\hat{\mathbf{i}} + \frac{\partial g}{\partial v}\hat{\mathbf{j}} = -u\sin v\hat{\mathbf{i}} + u\cos v\hat{\mathbf{j}}$$

Take $u = 1,\ v = \pi/4$

$$\mathbf{b}_1 = \frac{\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$$

$$\mathbf{b}_1 = -\frac{\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}}$$

For an orthogonal coordinate system,

$$\mathbf{b}_\mathrm{i} \cdot \mathbf{b}_\mathrm{j} = 0,\ \mathrm{i} \neq \mathrm{j}$$

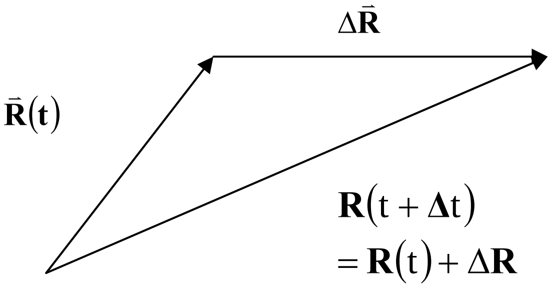
Consider the above example (polar coordinate), $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$

$\Rightarrow$ the polar coordinate is orthogonal and "the line of const. u" is normal to "the line of const. v" everywhere

● Curvilinear Coordinate System:

Consider a general coordinate transformation

$$x = f(u,v,w)$$
$$y = g(u,v,w)$$
$$z = h(u,v,w)$$



$$\lim_{\Delta t\to 0}\Delta\mathbf{R} = d\mathbf{R} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

Infinitesimal displacement vector $\lim_{\Delta t\to 0}\Delta\mathbf{R} = d\mathbf{R}$

$$\text{where}\begin{cases} dx = \dfrac{\partial f}{\partial u}du + \dfrac{\partial f}{\partial v}dv + \dfrac{\partial f}{\partial w}dw \\ dy = \dfrac{\partial g}{\partial u}du + \dfrac{\partial g}{\partial v}dv + \dfrac{\partial g}{\partial w}dw \\ dz = \dfrac{\partial h}{\partial u}du + \dfrac{\partial h}{\partial v}dv + \dfrac{\partial h}{\partial w}dw \end{cases}$$

$$\Rightarrow d\mathbf{R} = \left(\frac{\partial f}{\partial u}\hat{\mathbf{i}} + \frac{\partial g}{\partial u}\hat{\mathbf{j}} + \frac{\partial h}{\partial u}\hat{\mathbf{k}}\right)du + \left(\frac{\partial f}{\partial v}\hat{\mathbf{i}} + \frac{\partial g}{\partial v}\hat{\mathbf{j}} + \frac{\partial h}{\partial v}\hat{\mathbf{k}}\right)dv + \left(\frac{\partial f}{\partial w}\hat{\mathbf{i}} + \frac{\partial g}{\partial w}\hat{\mathbf{j}} + \frac{\partial h}{\partial w}\hat{\mathbf{k}}\right)dw \qquad (2.5)$$

or

$$d\mathbf{R} = \mathbf{b}_1 du + \mathbf{b}_2 dv + \mathbf{b}_3 dw \qquad \text{(curvilinear coordinate)}$$
$$= \hat{\mathbf{i}}\,dx + \hat{\mathbf{j}}\,dy + \hat{\mathbf{k}}\,dz \qquad \text{(Cartesian coordinate)}$$

Let $d\mathbf{R} = h_1\hat{\mathbf{e}}_1 du + h_2\hat{\mathbf{e}}_2 dv + h_3\hat{\mathbf{e}}_3 dw = \mathbf{b}_1 du + \mathbf{b}_2 dv + \mathbf{b}_3 dw$

where $\mathbf{b}_1$ is the natural basis vector $(i = 1,2,3)$ (2.6)

and $h_i = |\mathbf{b}_i|$ the length of the basis vector

A 3D vector equation can be decomposed into 3 scalar equations.

For example,

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \underbrace{\begin{pmatrix} \partial f/\partial u & \partial f/\partial v & \partial f/\partial w \\ \partial g/\partial u & \partial g/\partial v & \partial g/\partial w \\ \partial h/\partial u & \partial h/\partial v & \partial h/\partial w \end{pmatrix}}_{\text{the Jacobian matrix}} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} \qquad (2.7)$$

the determinant of Jacobian matrix = Jacobian

If determinant $= 0 \Rightarrow \vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ are on the same plane.

Consider the line integral along $d\mathbf{R}$

Definition: $ds = |d\mathbf{R}| = (d\mathbf{R} \cdot d\mathbf{R})^{1/2}$

In terms of curvilinear vectors,

$$d\mathbf{R} = \mathbf{b}_1 du + \mathbf{b}_2 dv \quad \text{(2D case only)}$$

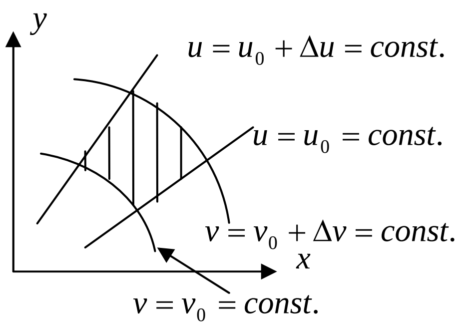
$$\Rightarrow ds = (d\mathbf{R} \cdot d\mathbf{R})^{1/2}$$

$$= \left[(\mathbf{b}_1 \cdot \mathbf{b}_1)(du)^2 + 2(\mathbf{b}_1 \cdot \mathbf{b}_2)du\,dv + (\mathbf{b}_2 \cdot \mathbf{b}_2)(dv)^2\right]^{1/2}$$

if $\mathbf{b}_1 \perp \mathbf{b}_2$ (orthogonal), $h_1 = |\mathbf{b}_1|$, $h_2 = |\mathbf{b}_2|$

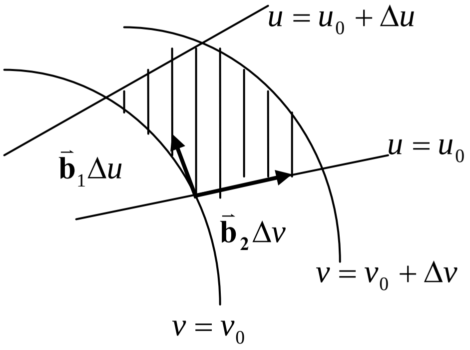
$$= \left[h_1^{\,2}(du)^2 + h_2^{\,2}(dv)^2\right]^{1/2} \quad \text{for orthogonal system}$$

Consider a surface integral along the 2D $d\vec{\mathbf{R}}$ space



To find $\iint_R g(u,v)dA = ?$

Consider $\lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta A = dA$



As $\Delta u \to 0, \Delta v \to 0, \Delta A \to$ parallelogram

$$dA = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta A = \left| \mathbf{b}_1 du \times \mathbf{b}_2 dv \right|$$

$$= \left| \mathbf{b}_1 \times \mathbf{b}_2 \right| dudv$$

for orthogonal coordinate, $\mathbf{b}_1 \perp \mathbf{b}_2$

$$= \left| \mathbf{b}_1 \right| \left| \mathbf{b}_2 \right| dudv$$

$$= h_1 h_2 dudv \qquad (2.8)$$

Consider a volume integral along the 3D $d\mathbf{R}$ space

$$dV = \left| \mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3) \right| dudvdw \qquad (2.9)$$

As $\Delta u \to 0, \Delta v \to 0, \Delta w \to 0$, $\Delta V \to$ parallelepiped

For orthogonal coordinate, $\mathbf{b}_1 \perp \mathbf{b}_2$, $\mathbf{b}_2 \perp \mathbf{b}_3$, $\mathbf{b}_3 \perp \mathbf{b}_1$

$$dV = h_1 h_2 h_3 dudvdw \qquad (2.10)$$

Ex: $\begin{cases} x = a\cosh u \cos v, & \text{where } \cosh u = \dfrac{e^u + e^{-u}}{2} \\ y = a\sinh u \sin v, & \text{where } \sinh u = \dfrac{e^u - e^{-u}}{2} \end{cases}$

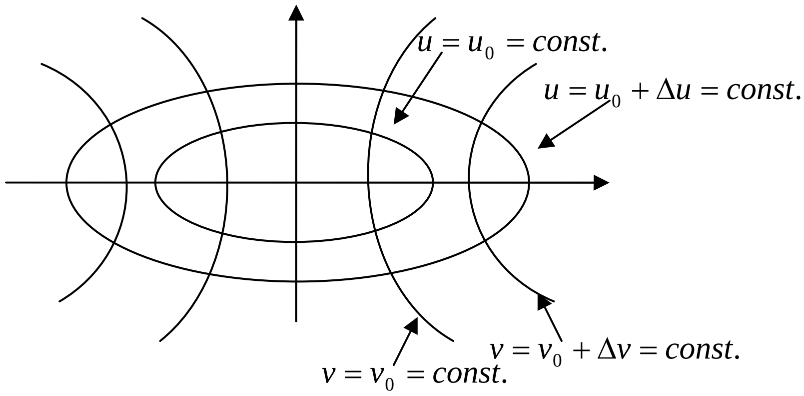
Sol: $\dfrac{x}{a\cosh u} = \cos v \quad (1), \quad \dfrac{y}{a\sinh u} = \sin v \qquad (2)$

$$\Rightarrow (1)^2 + (2)^2 : \ \cos^2 v + \sin^2 v = 1 = \frac{x^2}{a^2\cosh^2 u} + \frac{y^2}{a^2\sinh^2 u} \qquad \text{ellipse}$$

$$\frac{x}{a\cos v} = \cosh u \quad (3), \qquad \frac{y}{a\sin v} = \sinh u \qquad (4)$$

$$\Rightarrow (3)^2 - (4)^2 : \ \cosh^2 u - \sinh^2 u = 1 = \frac{x^2}{a^2\cosh^2 v} + \frac{y^2}{a^2\sin^2 v} \qquad \text{hyperbola}$$

For $\begin{cases} x = a\cosh u \cos v = f(u,v) \\ y = a\sinh u \sin v = g(u,v) \end{cases}$



Q: Whether $(u = u_0)$ & $(v = v_0)$ are orthogonal or not?

Sol :

$$\mathbf{b}_1 = \frac{\partial f}{\partial u}\hat{\mathbf{i}} + \frac{\partial g}{\partial u}\hat{\mathbf{j}} = a\sinh u \cos v\,\hat{\mathbf{i}} + a\cosh u \sin v\,\hat{\mathbf{j}}$$

$$\mathbf{b}_2 = \frac{\partial f}{\partial v}\hat{\mathbf{i}} + \frac{\partial g}{\partial v}\hat{\mathbf{j}} = -a\cosh u \sin v\,\hat{\mathbf{i}} + a\sinh u \cos v\,\hat{\mathbf{j}}$$

$$\Rightarrow \mathbf{b}_1 \cdot \mathbf{b}_2 = 0 \quad \therefore \mathbf{b}_1 \perp \mathbf{b}_2 \quad \text{orthogonal!}$$

And $$|\mathbf{b}_1| = \left[a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v\right]^{1/2}$$

$$= a\left[\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v\right]^{1/2}$$

$$= a\left[\sinh^2 u(1 - \sin^2 v) + (1 + \sinh^2 u)\sin^2 v\right]^{1/2}$$

$$= a\left[\sinh^2 u + \sin^2 v\right]^{1/2} = h_1$$

$$|\mathbf{b}_2| = \left[a^2 \cosh^2 u \sin^2 v + a^2 \sinh^2 u \cos^2 v\right]^{1/2}$$

$$= a\left[\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v\right]^{1/2}$$

$$= a\left[(1 + \sinh^2 u)\sin^2 v + \sinh^2 u(1 - \sin^2 v)\right]^{1/2}$$

$$= a\left[\sinh^2 u + \sin^2 v\right]^{1/2} = h_2$$