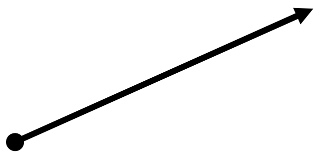
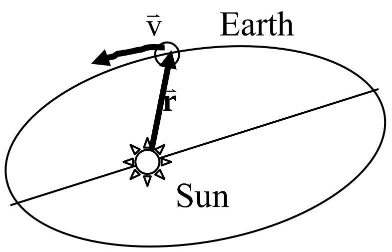
Chapter 1 Vector Algebra

Definition: Vector = A directed magnitude



Example:



$|\mathbf{r}|$ = the length or magnitude of displacement vector $\mathbf{r}$

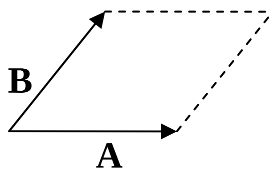
Scalar Product (Dot Product):

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\cdot|\mathbf{B}|\cos\theta = \mathbf{B}\cdot\mathbf{A} \qquad (1.1)$$

Cross Product:

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{e} \qquad (1.2)$$

where $\hat{e}$ = unit vector normal to both $\mathbf{A}$ and $\mathbf{B}$ (Right-Hand Rule)

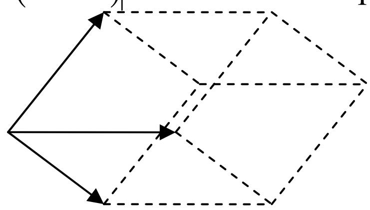


$|\mathbf{A}\times\mathbf{B}|$ = area of parallelogram spanned by $\mathbf{A}$ and $\mathbf{B}$

Scalar Triple Product:

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \qquad (1.3)$$

$|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ = volume of the parallelepiped spanned by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$

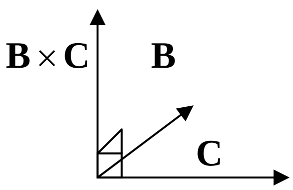


$\mathbf{V}$ = base area $\cdot$ height

$$\mathbf{V} = |\mathbf{A}\times\mathbf{B}| \cdot \text{projection of } \mathbf{C} \quad (\perp \text{ to } \mathbf{A} \text{ and } \mathbf{B})$$
$$= |(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}|\mathbf{C}\cdot\hat{e}$$

Vector Triple Product:

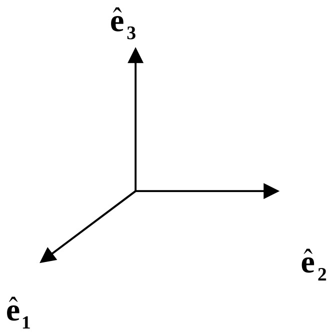
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \text{vector}\begin{cases}\perp \text{ to } \mathbf{A} \\ \perp \text{ to } (\mathbf{B}\times\mathbf{C})\end{cases}$$
$$= \alpha\mathbf{B} + \beta\mathbf{C}$$



Question: α = ? β = ?

Let $\hat{\mathbf{e}}_1$ be unit vector along $\mathbf{B}$

$\hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ both $\perp$ to $\hat{\mathbf{e}}_1$



$\Rightarrow \hat{\mathbf{e}}_2$ is in the plane of $\mathbf{B}$ and $\mathbf{C}$

$\hat{\mathbf{e}}_3$ is $\perp$ to this plane

$$(\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3) \Rightarrow \mathbf{C} = C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2$$
$$\mathbf{B} = B\hat{\mathbf{e}}_1$$
$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

Thus

$$\mathbf{B}\times\mathbf{C} = (B\hat{\mathbf{e}}_1)\times(C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2)$$
$$= BC_1\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_1 + BC_2\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_2$$
$$= BC_2\hat{\mathbf{e}}_3$$

$$
\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3)\times(BC_2\hat{\mathbf{e}}_3) \\
&= A_1BC_2\underbrace{\hat{\mathbf{e}}_1\times\hat{\mathbf{e}}_3}_{=-\hat{\mathbf{e}}_2} + A_2BC_2\underbrace{\hat{\mathbf{e}}_2\times\hat{\mathbf{e}}_3}_{=+\hat{\mathbf{e}}_1} + A_3BC_2\underbrace{\hat{\mathbf{e}}_3\times\hat{\mathbf{e}}_3}_{=\vec{\mathbf{0}}} \\
&= -A_1BC_2\hat{\mathbf{e}}_2 + A_2BC_2\hat{\mathbf{e}}_1
\end{aligned}
$$

add & subtract $A_1BC_1\hat{\mathbf{e}}_1$

$$
\begin{aligned}
&= A_1BC_1\hat{\mathbf{e}}_1 - A_1BC_2\hat{\mathbf{e}}_2 + A_2BC_2\hat{\mathbf{e}}_1 - A_1BC_1\hat{\mathbf{e}}_1 \\
&= (A_1C_1 + A_2C_2)\mathbf{B} - A_1B(C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2) \\
&= (A_1C_1 + A_2C_2)\mathbf{B} - A_1B\mathbf{C} \\
&= \alpha\mathbf{B} + \beta\mathbf{C} \\
&= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}
\end{aligned}
$$

$$\therefore \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad (1.4)$$

Reference: Speigel, M., 1968: Schaum's Outline of Vector Analysis, McGraw-Hill Company.

Derivatives of Vectors:

$\mathbf{A}(t) =$ vector $\mathbf{A}$, function of t

$$\frac{d\mathbf{A}}{dt}(t) = \text{vector} = \lim_{\Delta t\to 0}\frac{\mathbf{A}(t+\Delta t) - \mathbf{A}(t)}{\Delta t}$$

$$\frac{d}{dt}[\mathbf{A}(t)\cdot\mathbf{B}(t)] = \lim_{\Delta t\to 0}\frac{\mathbf{A}(t+\Delta t)\cdot\mathbf{B}(t+\Delta t) - \mathbf{A}(t)\cdot\mathbf{B}(t)}{\Delta t}$$

$$\Rightarrow \frac{d}{dt}[\mathbf{A}(t)\cdot\mathbf{B}(t)] = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt} \qquad (1.5) \qquad \text{(order is changeable)}$$

$$\frac{d}{dt}[\mathbf{A}(t)\times\mathbf{B}(t)] = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt} \qquad (1.6) \qquad \text{(order is not changeable)}$$